ELEMENTARY PRINCIPLES

OF

THE THEORIES OF ELECTRICITY, HEAT, AND MOLECULAR ACTIONS.

DESIGNED

FOR THE USE OF STUDENTS IN THE UNIVERSITY.

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PART I.

ON ELECTRICITY.

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INTRODUCTION.

M. Poisson in his Memoirs on Electricity, Magnetism and Molecular Actions, M. Ampere in his 'Theorie des Phenomenes Electro-dynamiques,' and Fourier in his 'Theorie de la Chaleur,' have been the respective founders of the physical sciences considered in this treatise in a mathematical point of view. The subject of electricity (including what is called ordinary electricity, Voltaic actions and magnetism,) forming in itself a complete system, is the sole object of the first part of this work, the other subjects being reserved for the Second Part; and as the ordinary course of mathematical reading in the University is a sufficient preparation for the study of the branches of science here treated, it is hoped that the suggestion recently made by a distinguished member of the University, will be in some degree answered in the present Treatise.*

As an acquaintance with the properties of the remarkable functions treated by Laplace in the Mec. Cel. Liv. III. is indispensable in investigations respecting electricity, instead of referring to that work I have here introduced them under the form of Preliminary Propositions; I have however followed a different rout, making the functions which shall possess those properties, the objects of investigation; and

* Whewell's Dynamics, 2nd Ed. Preface, p. XVIII.

have thus arrived at a more general class of functions (which are of great use in investigations relative to Latent Electricity,) and also obtained several new and remarkable theorems with respect to Laplace's functions: it must be added that on referring to Crelle's Journal, I found that M. Jacobi had anticipated me with a respect to few of the theorems alluded to.

It was natural to consider the manner in which electricity is disposed in bodies, previous to its becoming sensible by the action of electro-motive causes; this is the object of the second chapter, and I am not aware that it has beeen before made the subject of mathematical investigation.

It could answer no useful purpose to point out what is new in the remaining parts of the work; that will easily be recognised by those who are already acquainted with the subject, and those who are unacquainted would not benefit by the information; I shall only add that the sixth and seventh chapters contain the theories of Ampere on Voltaic actions, and Poisson on magnetism, with such modifications as seemed to simplify the processes employed by those writers.

I have to return my best thanks to Professor Cumming, for the facilities afforded me by the use of his apparatus, to confirm experimentally some of the results deduced in this work, from theoretical views.

R. M.

Caius College, *June*, 1833.

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ERRATA.

PAGE	LINE	ERROR.	CORRECTION.
35	11	ελεκτρον	ηλεκτρον
26	2 from bottom	negative	positive
37	2	positive	negative
53	2	origin	centre
54	15	a_2 a_3	$2a_2$ $3a_3$
—	17	a_1 a_2	$2a_1$ $3a_2$
85	2		add sphere
96	2	=	+
107	27	currents	elementary currents.

ON ELECTRICITY.

ON ELECTRICITY.

CHAPTER I.

PRELIMINARY PROPOSITIONS.

THE following propositions, belonging to Pure Mathematics, are of the greatest importance in the investigations relative to Electricity.

PROP. I.

To find a rational and entire function of given dimensions with respect to any variable, such that when multiplied by *any* rational and entire function of lower dimensions, the integral of the product taken between the limits 0 and 1 shall always vanish.

Let $f(t)$ be the required function of n dimensions with respect to the variable t; then the proposed condition will evidently require the following equations to be separately true; namely,

$$(a) \ldots\ldots \int_t f(t) = 0,\ \int_t f(t) \,.\, t = 0,\ \int_t f(t) \,.\, t^2 = 0, \ldots\ldots \int_t f(t) \,.\, t^{n-1} = 0$$

each integral being taken between the given limits.

Let the indefinite integral of $f(t)$, commencing when $t=0$, be represented by $f_1(t)$; the indefinite integral of $f_1(t)$, commencing also when $t=0$, by $f_2(t)$; and so on, until we arrive at the function $f_n(t)$, which is evidently of $2n$ dimensions, then the method of integrating by parts will give, generally,

$$\int_t f(t) \,.\, t^r = t^r f_1(t) - x t^{r-1} f_2(t) + x \,.\, (x-1) \,.\, t^{r-2} \,.\, f_3(t) - \&c.$$

Let us now put $t = 1$, and substitute for x the values 1, 2, 3, ... $(n-1)$ successively; then in virtue of the equations (a), we get,

$$(b) \ldots\ldots\ldots f_1(t) = 0, \quad f_2(t) = 0, \quad f_3(t) = 0, \ldots\ldots\ldots f_n(t) = 0.$$

Hence, the function $f_n(t)$ and its $(n-1)$ successive differential coefficients vanish, both when $t = 0$, and when $t = 1$; therefore t^n and $(1-t)^n$ are each factors of $f_n(t)$; and since this function is of $2n$ dimensions it admits of no other factor but a constant c.

Putting $1 - t = t'$, we thus obtain

$$f_n(t) = c \cdot (tt')^n;$$

$$\text{and therefore } f(t) = c \frac{d^n (tt')^n}{dt^n}.$$

Corollary. If we suppose the first term of $f(t)$, when arranged according to the powers of t, to be unity, we evidently have $c = \dfrac{1}{1 \cdot 2 \cdot 3 \ldots\ldots n}$; on this supposition we shall denote the above quantity by P_n.

Prop. II.

The function P_n which has been investigated in the preceding proposition, is the same as the coefficient of h^n in the expansion of the quantity

$$\{1 - 2h \cdot (1 - 2t) + h^2\}^{-\frac{1}{2}}.$$

Let u be a quantity which satisfies the equation

$$(c) \ldots\ldots\ldots\ldots u = t + h \cdot u (1 - u);$$

$$\text{that is, } u = -\frac{1-h}{2h} + \frac{1}{2h} \cdot \{1 - 2h \cdot (1 - 2t) + h^2\}^{\frac{1}{2}};$$

$$\text{therefore } \frac{du}{dt} = \{1 - 2h (1 - 2t) + h^2\}^{-\frac{1}{2}}.$$

But if as before we write t' for $1-t$, we have by Lagrange's Theorem applied to the equation (c)

$$u = t + h \cdot tt' + \frac{h^2}{1 \cdot 2} \cdot \frac{d(tt')^2}{dt} + \frac{h^3}{1 \cdot 2 \cdot 3} \frac{d^2 \cdot (tt')^3}{dt^2} + \text{\&c.}$$

If we differentiate, and put for $\frac{d^n(tt')^n}{dt^n}$ its value $1 \cdot 2 \cdot 3 \ldots n \cdot P_n$ given by the former proposition, we get

$$\frac{du}{dt} = 1 + P_1 h + P_2 h^2 + P_3 h^3 + \text{\&c.}$$

Comparing this with the above value of $\frac{du}{dt}$ the proposition is manifest.

Prop. III.

To integrate $P_n \cdot t^x$ from $t=0$ to $t=1$; x being any quantity entire or fractional between -1 and $+\infty$.

Since P_n is a function of n dimensions of which the first term is unity, we may represent it by

$$\ldots\ldots 1 + A_1 t + A_2 t^2 + A_3 t^3 + \ldots\ldots + A_n t^n.$$

Hence, between the above limits we have

$$(d) \ldots\ldots \textstyle\int_t P_n t^x = \frac{1}{x+1} + \frac{A_1}{x+2} + \frac{A_2}{x+3} + \ldots\ldots + \frac{A_n}{x+n+1}.$$

The actual sum of the terms composing the right-hand member of this equation is a fraction, of which the denominator is $(x+1)(x+2)(x+3)\ldots\ldots(x+n+1)$ and the numerator is some function of x of n dimensions, but which, by the nature of P_n, ought to vanish when we make x successively equal to $0, 1, 2, 3 \ldots\ldots (n-1)$, and therefore can be no other than $c \cdot x(x-1)(x-2)\ldots\ldots(x-n+1)$, c being a quantity depending on n only: the right-hand member of the equation (d) is therefore equivalent to the fraction

$$c \cdot \frac{x \cdot (x-1)(x-2)\ldots\ldots(x-n+1)}{(x+1)(x+2)(x+3)\ldots\ldots(x+n+1)}.$$

If we multiply this quantity by $x+1$ and then put $x=-1$ it becomes $c(-1)^n$, but the right-hand member of the equation (d) under the same circumstances reduces itself to unity; hence $c=(-1)^n$ substituting this value we get the required integral, namely, $(-1)^n \cdot \frac{x.(x-1)(x-2)......(x-n+1)}{(x+1)(x+2)(x+3)......(x+n+1)}$.

Prop. IV.

To determine, between the limits $t=0$ and $t=1$, the integral of the product $P_m . P_n$.

Case (1). When m and n are *unequal*, one of them as n must be the greater; then P_m is a function of lower dimensions than P_n, and therefore by Prop. I. the required integral is zero.

Case (2). When m and n are *equal*, then since

$$P_n = \frac{d^n (tt')^n}{1.2.3...ndt^n};$$

put $1-t$ for t', that is, put

$$(-1)^n . \{t^n - nt^{n-1} + \frac{n.(n-1)}{1.2} . t^{n-2} - \&c.\}$$

instead of t'^n, then multiplying by t^n and taking actually the n^{th} differential coefficient, we get

$$(e)...P_n = (-1)^n . \frac{2n(2n-1)(2n-2)...(n+1)}{1.2.3...n} . t^n + Bt^{n-1} + Ct^{n-2} \&c.$$

where B, C, &c. represent constants, of which it is unnecessary to calculate the values.

Multiply both sides of the equation (e) by P_n, and observing that between the assigned limits we have $\int_t P_n t^{n-1} = 0$, $\int_t P_n t^{n-2} = 0$ &c., we get

$$\int_t P_n . P_n = (-1)^n . \frac{2n(2n-1)(2n-2)......(n+1)}{1.2.3......n} . \int_t P_n t^n,$$

but by Prop. III.

$$\int_t P_n t^n = (-1)^n \cdot \frac{n.(n-1)(n-2)\ldots\ldots 2.1}{(n+1)(n+2)(n+3\ldots\ldots(2n+1)}.$$

Hence we have $\int_t P_n \cdot P_n = \dfrac{1}{2n+1}$.

PROP. V.

To develop the function P_n.

First Expansion. By Prop. I., we have,

$$P_n = \frac{1}{1.2.3\ldots\ldots n} \cdot \frac{d^n.(tt')^n}{dt^n}.$$

Hence, $P_n = \dfrac{1}{1.2.3\ldots n} \cdot \dfrac{d^n}{dt^n} \left\{ t^n - n t^{n+1} + \dfrac{n.(n-1)}{1.2} . t^{n+2} - \&c. \right\}$

$$(e)\ldots\ldots = 1 - \frac{n}{1} . \frac{n+1}{1} . t + \frac{n.(n-1)}{1.2} . \frac{(n+1)(n+2)}{1.2} . t^2 - \&c.$$

Second Expansion. If u and v are functions of any variable t, then the theorem of Leibnitz gives the indentity

$$\frac{d^n(uv)}{dt^n} = v\frac{d^n u}{dt^n} + n\frac{dv}{dt} . \frac{d^{n-1}u}{dt^{n-1}} + \frac{n(n-1)}{1.2} . \frac{d^2 v}{dt^2} . \frac{d^{n-2}v}{dt^{n-2}} + \&c.$$

put $u = t^n$ and $v = t'^n$ and dividing by $1.2.3\ldots n$ we have,

$$(f)\ldots\ldots P_n = t'^n - \left(\frac{n}{1}\right)^2 t'^{n-1} t$$

$$+ \left\{\frac{n(n-1)}{1.2}\right\}^2 t'^{n-2} t^2 - \left\{\frac{n(n-1)(n-2)}{1.2.3}\right\}^2 t'^{n-3} t^3 + \&c.$$

Third Expansion. Put $1 - 2t = \mu$ and therefore $tt' = \dfrac{1-\mu^2}{2^2}$,

hence $P_n = \dfrac{1}{2^n} . \dfrac{1}{1.2.3\ldots n} . \dfrac{d^n(\mu^2-1)^n}{d\mu^n}$

$$= \frac{1}{2.4.6\ldots 2n} . \frac{d^n}{d\mu^n} . \left\{ \mu^{2n} - n\mu^{2n-2} + \frac{n.(n-1)}{1.2} . \mu^{2n-4} - \&c. \right\}$$

$$(g)\ldots\ldots = \frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}.\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}.\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}.\mu^{n-4} - \&c.\right\}$$

Fourth Expansion. Put $1-2t=\cos\theta$; hence by Prop. II. we have, P_n = coefficient of h^n in $(1-2h\cos\theta+h^2)^{-\frac{1}{2}}$.

But $(1-2h\cos\theta+h^2)^{-\frac{1}{2}} = (1-h\epsilon^{\theta\sqrt{-1}})^{-\frac{1}{2}}.(1-h\epsilon^{-\theta\sqrt{-1}})^{-\frac{1}{2}}$,

$$= \left\{1+\tfrac{1}{2}.h\epsilon^{\theta\sqrt{-1}} + \frac{1.3}{2.4}.h^2\epsilon^{2\theta\sqrt{-1}} + \&c.\right\} \times \left\{1+\tfrac{1}{2}h\epsilon^{-\theta\sqrt{-1}} + \frac{1.3}{2.4}.h^2\epsilon^{-2\theta\sqrt{-1}} + \&c.\right\}$$

and if we collect the coefficient of h^n in this product, and substitute trigonometrical expressions for exponential, we get,

$$(h)\ldots\ldots P_n = 2.\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}.\left\{\cos n\theta + \frac{1.n}{1.(2n-1)}.\cos(n-2)\theta + \frac{1.3.n(n-1)}{1.2.(2n-1)(2n-3)}\cos(n-4).\theta + \&c.\right\}$$

Corollary. The coefficient of $\cos(k\theta)$ in P_n is zero, when $n-k$ is odd, but when $n-k$ is even as $2i$, its value is then

$$2.\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}.\frac{1.3\ldots(2i-1)}{1.2\ldots i}.\frac{n(n-1)\ldots(n-i+1)}{(2n-1)(2n-3)\ldots(2n-2i+1)}.$$

But $$\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i} = 2^i.\frac{1.3.5\ldots(n-k-1)}{2.4.6\ldots(n-k)},$$

and $$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}.\frac{n(n-1)\ldots(n-i+1)}{(2n-1)(2n-3)\ldots(2n-2i+1)}$$

$$= 2^{-i}.\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}.\frac{2n(2n-2)\ldots(n+k+2)}{(2n-1)(2n-3)\ldots(n+k+1)}$$

$$= 2^{-i}.\frac{1.3.5\ldots(n+k-1)}{2.4.6\ldots(n+k)}.$$

Hence the coefficient of $\cos(k\theta)$ in P_n when $n-k$ is even is

$$2\cdot\frac{1.3.5\ldots(n-k-1)}{2.4.6\ldots(n-k)}\cdot\frac{1.3.5\ldots(n+k-1)}{2.4.6\ldots(n+k)}.$$

This quantity is evidently the coefficient of $h^n\cos k\theta$ in $(1-2h\cos\theta+h^2)^{-\frac{1}{2}}$ when $n-k$ is even, and a similar process will shew that the coefficient of $h^{n-1}\cos k\theta$, in $(1-2h\cos\theta+h^2)^{-\frac{3}{2}}$ is zero when $n-k$ is even, but when $n-k$ is odd its value is

$$2\cdot\frac{1.3.5\ldots(n+k)}{2.4.6\ldots(n+k-1)}\cdot\frac{1.3.5\ldots(n-k)}{2.4.6\ldots(n-k-1)}.$$

PROP. VI.

To expand a given function of t, in terms of functions of the same nature as P_n, when such an expansion is possible.

Let y be the given function of t, and suppose

$$y = aP_0 + bP_1 + cP_2 + \&c.,$$

where a, b, c, &c. are constant coefficients to be determined.

Multiply by P_0 and integrate from $t=0$ to $t=1$; hence if the definite integral $\int_t P_0 y$ be represented by α, we have

$$\begin{aligned}\alpha &= a\int_t P_0P_0 + b\int_t P_0P_1 + c\int_t P_0P_2 + \&c.\\ &= a \text{ by Prop. IV.}\end{aligned}$$

Again, multiply the same identity by P_1 and integrate from $t=0$ to $t=1$, representing the definite integral $\int_t P_1 y$ by β, hence

$$\begin{aligned}\beta &= a\int_t P_1P_0 + b\int_t P_1P_1 + c\int_t P_1P_2 + \&c.\\ &= \frac{b}{3} \text{ by Prop. IV.}\end{aligned}$$

Similarly, if we put the definite integral $\int_t P_2 y = \gamma$, we get

$$\gamma = \frac{c}{5},$$

&c.

Thus a, b, c, &c. are known and substituting their values, we get

$$y = \alpha P_0 + 3\beta P_1 + 5\gamma P_2 + 7\delta P_3 + \text{\&c.}$$

where α, β, γ, &c. are all numerical quantities, and P_0, P_1, P_2, &c. rational and entire functions of t, the dimensions of which are respectively expressed by their sub-indices.

Note. In the preceding investigation it has been assumed that the function y was capable of being represented by a series of the form $aP_0 + bP_1 + cP_2 + \text{\&c.}$, we shall here consider in what cases such an identity is possible.

First, when y is any rational and entire function of t, then this identity is evidently possible, for if y be of m dimensions it has $m + 1$ coefficients in its most general form; if therefore we take the first $m + 1$ terms of the above series, they express another function of t of m dimensions containing $m + 1$ arbitrary constants; and if we equate like powers of t we shall have $m + 1$ equations by which these constants may be determined.

EXAMPLE. To expand t^m, where m is a positive integer in functions of the same nature as P_n.

Here by Prop. III. we have

$$\alpha = \int_t P_0 t^m = \frac{1}{m+1},$$

$$\beta = \int_t P_1 t^m = \frac{m}{(m+1)(m+2)},$$

$$\gamma = \int_t P_2 t^m = \frac{m\,.\,(m-1)}{(m+1)(m+2)(m+3)}.$$

$$\text{\&c.} = \text{\&c.}$$

Hence,

$$t^m = \frac{1}{m+1}\,.\,P_0 + \frac{3m}{(m+1)(m+2)}\,.\,P_1 + \frac{5\,.\,m\,.\,(m-1)}{(m+1)(m+2)(m+3)}\,.\,P_2 + \text{\&c.}$$

Secondly, when y is any transcendent, which may be expressed in a converging series always finite, from $t=0$ to $t=1$; for we may then consider it as consisting of a great but finite number of terms, and therefore the former reasoning will apply; and it is easy to see that none of the quantities α, β, γ, &c. will be infinite in this case*, the series in the required form will moreover be converging, for values of t between 0 and 1.

Example. To expand t'^m where $t' = 1 - t$.

Let P'_0, P'_1, P'_2, &c. denote the values of P_0, P_1, P_2, &c. when t' is put for t; it is evident by Prop. I, that $P'_n = (-1)^n . P_n$.

$$\begin{aligned}\text{Hence we have } \alpha &= \textstyle\int_t P_0 t'^m \text{ from } t=0 \text{ to } t=1\\ &= \textstyle\int_{t'} P'_0 t'^m \text{ from } t'=0 \text{ to } t'=1\\ &= \frac{1}{m+1} \text{ since the accents may be omitted.}\end{aligned}$$

$$\begin{aligned}\text{Similarly, } \beta &= \textstyle\int_t P_1 t'^m \text{ from } t=0 \text{ to } t=1\\ &= -\textstyle\int_{t'} P'_1 t'^m \text{ from } t'=0 \text{ to } t'=1\\ &= -\frac{m}{(m+1)(m+2)},\\ \text{\&c.} &= \text{\&c.}\end{aligned}$$

Therefore

$$t'^m = \frac{1}{m+1} . P_0 - \frac{3m}{(m+1)(m+2)} . P_1 + \frac{5m . (m-1)}{(m+1)(m+2)(m+3)} - \text{\&c.}$$

which holds true for any value of m from -1 to $+\infty$, supposing t to be kept within the limits 0 and 1.

Corollary. Since $P_n = (-1)^n . P'_n$, this formula becomes

$$t'^m = \frac{1}{m+1} . P'_0 + \frac{3m}{(m+1) . (m+2)} . P'_1 + \frac{5m . (m-1)}{(m+1)(m+2)(m+3)} . P'_2 + \text{\&c.}$$

* Vide Camb. Trans. Vol. IV. p. 359. Art. 3, &c.

and if we now omit the accents, we get the same expression as that already obtained for t^m, which is therefore true for *all* values of m, from -1 to $+\infty$, provided t is between 0 and 1.

Thus, let $m = -\frac{1}{2}$, we get

$$\frac{1}{\sqrt{t}} = 2\left\{P_0 + P_1 + P_2 + \&c.\right\}$$

which may be confirmed by observing that by Prop. II. we have

$$\left\{1 - 2h(1 - 2t) + h^2\right\}^{-\frac{1}{2}} = P_0 + P_1 h + P_2 h^2 + \&c.$$

and then putting $h = 1$.

Again, if we differentiate the general expression for t^m, and then put $m = 0$, we get

$$-\log.(t) = (P_0 + P_1) + \tfrac{1}{2}(P_1 + P_2) + \tfrac{1}{3}(P_2 + P_3) + \&c.$$

A simple numerical illustration may be given by putting $t = \frac{1}{2}$ when we have $P_{2n} =$ coefficient of h^{2n} in $(1 + h^2)^{-\frac{1}{2}}$ by Prop. II.

$$= (-1)^n \cdot \frac{1.3.5\ldots 2n-1}{2.4..6\ldots 2n},$$

and $P_{2n+1} = 0$;

$$\therefore \frac{1}{2\sqrt{(\frac{1}{2})}} = \sqrt{(\tfrac{1}{2})} = 1 - \tfrac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \&c.$$

$$\text{and } -\log.(\tfrac{1}{2}) = \log.(2) = 1 - \frac{5}{2.3}\cdot\tfrac{1}{2} + \frac{9}{4.5}\cdot\frac{1.3}{2.4} - \&c.$$

Prop. VII.

To compare the *indefinite* integral of P_n of any order not higher than the n^{th}, and commencing when $t = 0$ with its differential coefficient of the same order.

If we denote the m^{th} integral of P_n by the symbol $\int_t^m P_n$

$$\text{then, since } P_n = \frac{1}{1.2.3\ldots n}\cdot\frac{d^n(tt')^n}{dt^n},$$

we have $\int_t^m P_n = \dfrac{1}{1.2.3\ldots n} \cdot \dfrac{d^{n-m}(tt')^n}{dt^{n-m}}$,

all the successive integrals commencing when $t = 0$.

Expanding the right-hand member by the Theorem of Leibnitz, we have

$$1.2.3\ldots n\int_t^m P_n = t'^n . \frac{d^{n-m} . t^n}{dt^{n-m}} + (n-m)\frac{d . t'^n}{dt} . \frac{d^{n-m-1} . t^n}{dt^{n-m-1}}$$

$$+ \frac{(n-m)(n-m-1)}{1.2} . \frac{d^2 . t'^n}{dt^2} . \frac{d^{n-m-2} . t^n}{dt^{n-m-2}} + \&c.$$

and actually performing the differentiations, this expression becomes

$$n.(n-1)(n-2)\ldots(m+1).(tt')^m . \left\{t'^{n-m} - \frac{n}{1} . \frac{n-m}{m+1} . tt'^{n-m-1}\right.$$

$$\left. + \frac{n.(n-1)}{1.2} . \frac{(n-m)(n-m-1)}{(m+1)(m+2)} . t^2 t'^{n-m-2} - \&c.\right\}$$

In like manner, we have $\dfrac{d^m P_n}{dt^m} = \dfrac{1}{1.2.3\ldots n} . \dfrac{d^{n+m}(tt')^n}{dt^n}$, whence

$$1.2.3\ldots n\frac{d^m P_n}{dt^m} = t'^n . \frac{d^{n+m} . t^n}{dt^{n+m}} + (n+m)\frac{dt'^n}{dt} . \frac{d^{n+m-1} . t^n}{dt^{n+m-1}}$$

$$+ \frac{(n+m)(n+m-1)}{1.2} . \frac{d^2 . t'^n}{dt^2} . \frac{d^{n+m-2} . t^n}{dt^{n+m-2}} + \&c.$$

and performing the differentiations, the first m terms will vanish, and the expression will become

$$(-1)^m . n(n-1)(n-2)\ldots(m+1) \times (n+m)(n+m-1)\ldots(n+1)$$
$$\times n.(n-1)\ldots(n-m+1)$$

$$\times \left\{t'^{n-m} - \frac{n}{1}\,\frac{n+m}{m+1} . t'^{n-m-1}t\right.$$

$$\left. + \frac{n.(n-1)}{1.2}\,\frac{(n+m)(n+m-1)}{(m+1)(m+2)}\,t'^{n-m-1}t^2\ \&c.\right\}$$

Comparing both expressions, we get, finally

$$\int_t^m . P_n = \frac{(-tt')^m}{(n+m)(n+m-1)\ldots(n-m+1)} \cdot \frac{d^m . P_n}{dt^m}.$$

Corollary 1. Let $m = 1$, we get

$$\int_t P_n = -\frac{tt'}{n.(n+1)} \cdot \frac{dP_n}{dt},$$

from whence it is obvious that the definite integral of P_n taken between any two values of t which render P_n a maximum or a minimum, or from any such value of t to $t = 0$ or 1, always vanishes*.

Corollary 2. Put $1 - 2t = \mu$ in the equation just obtained, then since $tt' = \frac{1-\mu^2}{4}$, we get

$$\int_\mu P_n = \frac{(\mu^2 - 1)}{n.(n+1)} \cdot \frac{dP_n}{d\mu},$$

$$\text{hence } \frac{d}{d\mu}\left\{(1-\mu^2) \cdot \frac{dP_n}{d\mu}\right\} + n.(n+1).P_n = 0,$$

where P_n is expressed by a differential equation.

Corollary 3. We may also by this proposition integrate from $t = 0$ to $t = 1$ the quantity $(tt')^r . \frac{d^r P_m}{dt^r} . \frac{d^r P_n}{dt^r}$; for if we integrate by parts, observing that the part outside the sign of integration vanishes between limits, we get

$$\int_t (tt')^r . \frac{d^r P_m}{dt^r} . \frac{d^r P_n}{dt^r} = -\int_t \frac{d^{r-1} P_n}{dt^{r-1}} . \frac{d}{dt}\left\{(tt')^r . \frac{d^r P_m}{dt^r}\right\}.$$

But

$$\frac{d}{dt}\left\{(tt')^r \frac{d^r P_m}{dt^r}\right\} = (-1)^r . (m+r)(m+r-1)\ldots(m-r+1) . \int_t^{r-1} P_m$$

$$= -(m+r)(m-r+1)(tt')^{r-1} . \frac{d^{r-1} . P_m}{dt^{r-1}}.$$

* Camb. Trans. Vol. IV. p. 393.

Hence

$$\int_t (tt')^r \cdot \frac{d^r P_m}{dt^r} \cdot \frac{d^r P_n}{dt^r} = (m+r)(m-r+1) \int_t (tt')^{r-1} \cdot \frac{d^{r-1} \cdot P_m}{dt^{r-1}} \frac{d^{r-1} \cdot P_n}{dt^{r-1}},$$

we may thus reduce the index r by unity, and continuing the process, we should finally arrive at

$$(m+r)(m+r-1) \ldots (m-r+1) \cdot \int_t P_m P_n,$$

whence by Prop. IV. the required integral when m and n are unequal is zero, but when equal it is

$$\frac{1}{2m+1}(m+r)(m+r-1) \ldots (m-r+1).$$

PROP. VIII.

If Q_n be the coefficient of h^n, in the expansion of the quantity $\{1 - 2h(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') + h^2\}^{-\frac{1}{2}}$; also if γ represent $\cos\theta$, and $\mu = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, and F be any function of μ; then shall $\int_\gamma \int_\phi F \cdot Q_n = +2\pi \int_\mu F \cdot P_n$, the limits of μ being -1 and $+1$; of γ, the same; and of ϕ, 0 and 2π.

Conceive a fixed line drawn through the centre C of a sphere, of which the radius is unity.

Let a point P' be taken within the sphere, at a distance h from its centre; the straight line h making an angle θ' with the fixed line.

Let a point P be taken on the surface, and let the radius passing through it make an angle θ with the same fixed line; and let the planes of the angles θ θ' be inclined to a fixed plane at the angles ϕ and ϕ'.

Then μ is evidently the cosine of the angle (ω) included by the right lines CP and CP'.

Hence if F represent the density at the point P, then the sum of all the elements divided by their distances from P' is

expressed by $\int_\gamma \int_\phi \frac{F}{\delta}$; putting the distance $PP' = \delta$, and the double integral being extended over the surface of the sphere; the limits of ϕ being evidently 0 and 2π and of γ or $\cos\theta$, -1 and $+1$.

But since F is a function of γ, the annulus which contains all the points similarly situated as P with respect to P', will be expressed in mass by $-2\pi F \sin\omega . \delta\omega$ or $2\pi F \delta\mu$; the sum of all the elements of the spherical surface divided by their respective distances from P' will be now expressed by $\int_\mu \frac{2\pi F}{\delta}$, from $\mu = -1$ to $\mu = +1$.

$$\text{Hence} \quad \int_\gamma \int_\phi \frac{F}{\delta} = +2\pi \int_\mu \frac{F}{\delta}.$$

$$\text{But} \quad \frac{1}{\delta} = Q_0 + Q_1 h + Q_2 h^2 + \ldots \&\text{c.}$$

and also $= P_0 + P_1 h + P_2 h^2 + \ldots$ &c. where $P_n =$ the coefficient of h^n in $\{1 - 2h . (1 - 2t) + h^2\}^{-\frac{1}{2}}$ when $1 - 2t = \mu$;

and if we equate like powers of h, we have

$$\int_\gamma \int_\phi F Q_n = +2\pi \int_\mu F P_n.$$

Corollary. If $F = Q_m$ we obtain a theorem analogous to that given in Prop. IV. viz.

$$\int_\gamma \int_\phi Q_m Q_n = 2\pi \int_\mu P_m P_n = 0, \ m \text{ and } n \text{ being unequal,}$$

$$\int_\gamma \int_\phi Q_n . Q_n = +2\pi \int_\mu P_n . P_n = \frac{4\pi}{2n+1}.$$

Prop. IX.

The function Q_n satisfies the equation,

$$\frac{d}{d\gamma}\left\{(1-\gamma^2) . \frac{dQ_n}{d\gamma}\right\} + \frac{1}{1-\gamma^2} . \frac{d^2 Q_n}{d\phi^2} + n . (n+1) Q_n = 0.$$

Put as before $\mu = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, then if u be any function of θ and ϕ, the principles of the differential calculus give the equations

$$\frac{du}{d\gamma} = -\frac{1}{\sin\theta}\cdot\frac{du}{d\theta}.$$

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{du}{d\gamma}\right\} = \frac{d^2u}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\cdot\frac{du}{d\theta}.$$

But $\dfrac{du}{d\theta} = \dfrac{du}{d\mu}\cdot\dfrac{d\mu}{d\theta}$

$$= \frac{du}{d\mu}\{\cos\theta\sin\theta'\cos(\phi-\phi') - \cos\theta'\sin\theta\};$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{d^2u}{d\mu^2}\{\cos\theta\sin\theta'\cos(\phi-\phi')\cos\theta'\sin\theta\}^2 - \mu\frac{du}{d\mu};$$

and by a similar process we get

$$\frac{du}{d\phi} = -\frac{du}{d\mu}\{\sin\theta\sin\theta'\sin(\phi-\phi')\},$$

$$\frac{d^2u}{d\phi^2} = \frac{d^2u}{d\mu^2}.\sin^2\theta.\sin^2\theta'\sin^2(\phi-\phi') - \frac{du}{d\mu}.\sin\theta\sin\theta'\cos(\phi-\phi').$$

Now the substitution of these values in

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\cdot\frac{du}{d\gamma}\right\} + \frac{1}{\sin^2\theta}\cdot\frac{d^2u}{d\phi^2}$$

will reduce it to

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu},$$

$$\text{or, } \frac{d}{d\mu}\left\{(1-\mu^2)\cdot\frac{du}{d\mu}\right\}.$$

Assign now to u the particular value $Q_n = P_n$ as was shewn in the last proposition by putting $\gamma = 1 - 2t$, thence we obtain

$$\frac{d}{d\mu}\left\{(1-\mu^2)\cdot\frac{du}{d\mu}\right\} \text{ that is, } \frac{d}{dt}\left(tt'\frac{dP_n}{dt}\right),$$

$$= \frac{d}{d\gamma}\left\{(1-\gamma^2) \cdot \frac{dQ_n}{d\gamma}\right\} + \frac{1}{1-\gamma^2}\frac{d^2 Q_n}{d\phi^2} = \frac{d}{dt}\left(tt'\frac{dP_n}{dt}\right)$$

$$= -n \cdot (n+1) \cdot P_n$$

$$= -n(n+1) \cdot Q_n.$$

Prop. X.

To expand Q_n in a series arranged according to the cosines of the multiples of $\phi - \phi'$.

The term which involves $\cos k(\phi - \phi')$ in the expansion of the surd

$$\{1 - 2h[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')] + h^2\}^{-\frac{1}{2}}$$

can come from those terms only in which $\cos(\phi - \phi')$ is raised to powers equal to k, $k+2$, $k+4$, &c.; it must therefore contain the factor $\sin^k(\theta)$ or $(1-\gamma^2)^{\frac{k}{2}}$.

Now Q_n is the coefficient of h^n in the above-mentioned expansion, from whence it is easily seen that the coefficient of $\cos k(\phi - \phi')$ in Q_n is of the form

$$(1-\gamma^2)^{\frac{k}{2}} \cdot (a_0\gamma^{n-k} + a_1\gamma^{n-k-2} + a_2\gamma^{n-k-4} + \ldots),$$

and therefore, if this quantity be represented by q_k, we have

$$Q_n = q_0 + q_1\cos(\phi - \phi') + q_2\cos 2(\phi - \phi') + q_3\cos 3(\phi - \phi') + \text{\&c.}$$

Substitute this value for Q_n in the equation given in Prop. IX. and equate to zero the coefficient of $\cos k(\phi - \phi')$ which results, hence

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{dq_k}{d\gamma}\right\} + \frac{k^2}{1-\gamma^2} \cdot q_k + n(n+1)q_k = 0;$$

put now for q_k its value given above, and equating like powers of γ, we have in general

$$a_m = -\frac{(n-k-2m+1)(n-k-2m+2)}{2m(2n-2m+1)} \cdot a_{m-1},$$

from whence by assigning to m successive values, it follows that

$$q_k = a_0 (1-\gamma^2)^{\frac{k}{2}} \left\{\gamma^{n-k} - \frac{(n-k)(n-k-1)}{2 \cdot (2n-1)} \cdot \gamma^{n-k-2}\right.$$

$$\left. + \frac{(n-k)(n-k-1)(n-k-2)(n-k-3)}{2 \cdot 4 \cdot (2n-1) \cdot (2n-3)} \gamma^{n-k-4} - \&c.\right.$$

which equation may for abridgement be written

$$q_k = a_0 \cdot F_k(\gamma),$$

where a_0 remains yet to be determined.

Now, since θ and θ' are similarly involved in Q_n and therefore also in q_k, if we make $\cos\theta' = \gamma'$ it is evident that we must have

$$q_k = A \cdot F_k(\gamma') \cdot F_k(\gamma),$$

where A is independent of θ and θ'.

Put $\theta = \theta' = \dfrac{\pi}{2}$, and therefore $\gamma = \gamma' = 0$; then if $n-k$ is even, we have from the general value of $F_k(\gamma)$,

$$F_k(0) = \frac{1 \cdot 2 \cdot 3 \ldots\ldots\ldots (n-k)}{2 \cdot 4 \ldots (n-k) \times (2n-1)(2n-3)\ldots(n+k+1)},$$

and q_k, which in this case is the coefficient of $h^n \cos k(\phi - \phi')$ in the expansion of $\{1 - 2h\cos(\phi-\phi') + h^2\}^{-\frac{1}{2}}$ is expressed by $A\{F_k(0)\}^2$.

But by Prop. v. the same coefficient was shewn to be equal to

$$2 \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-k-1) \times 1 \cdot 3 \cdot 5 \ldots (n+k-1)}{2 \cdot 4 \cdot 6 \ldots (n+k) \times 2 \cdot 4 \cdot 6 \ldots (n+k)}$$

$$A = 2\left\{\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots n}\right\}^2 \cdot \frac{n \cdot (n-1) \ldots (n-k+1)}{(n+1)(n+2)\ldots(n+k)},$$

except when $k = 0$, for then we must take only half the value given by this formula.

But when $n-k$ is odd, then if we denote $\frac{d}{d\gamma}\{F_k(\gamma)\}$ by $F'_k(\gamma)$, we have

$$F'_k(0)=\frac{1.2.3......(n-k)}{2.4...(n-k-1)\times(2n-1)(2n-3)...(n+k+2)},$$

and it is evident that the coefficient of $\gamma\gamma'$ in q_k, is $A\{F'_k(0)\}^2$.

Now the surd of which $Q_n h^n$ is the general term when expanded, neglecting the second and higher powers of γ, γ', is evidently

$$\{1-2h\cos(\phi-\phi')+h^2\}^{-\frac{1}{2}}+h\gamma\gamma'\{1-2h\cos(\phi-\phi')+h^2\}^{-\frac{3}{2}},$$

from whence the coefficient of $\gamma\gamma'$ in q_k, is the same as the coefficient of h^{n-1} in

$$\{1-2h\cos(\phi-\phi')+h^2\}^{-\frac{3}{2}};$$

that is, by Prop. v.

$$2.\frac{1.3.5...(n-k)\times1.3.5...(n+k)}{2.4.6...(n-k-1)\times2.4.6...(n+k-1)};$$

comparing this with the value above obtained for the same, we get exactly the same value for A, as in the former case.

Hence the required development is

$$Q_n=\left\{\frac{1.3.5...(2n-1)}{1.2.3...n}\right\}^2.\{F_0(\gamma)F_0(\gamma')$$

$$+\frac{n}{n+1}.F_1(\gamma)F_1(\gamma')\cos(\phi-\phi')$$

$$+\frac{n(n-1)}{(n+1)(n+2)}.F_2(\gamma)F_2(\gamma')\cos2(\phi-\phi')+\&c.\}$$

Prop. XI.

To find the most general, rational and entire function, with respect to $\sin\theta$, $\cos\theta$, $\sin\phi$, $\cos\phi$, which will satisfy the equation

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\cdot\frac{du}{d\gamma}\right\}+\frac{1}{(1-\gamma^2)}\cdot\frac{d^2u}{d\phi^2}+n\cdot(n+1)\cdot u=0,$$

where γ is put for $\cos\theta$.

Suppose the required function is expressed in terms of the sines and cosines of the multiples of ϕ, and let the general term of this expansion be represented by $A_k \cos k\phi + B_k \sin k\phi$.

Substitute this value for u in the given equation, and comparing the terms which contain the same multiples of ϕ, we obtain, as in the last proposition,

$$A_k = a_k (1-\gamma^2)^{\frac{k}{2}} \left\{\gamma^{n-k} - \frac{(n-k)(n-k-1)}{2(2n-1)}\cdot\gamma^{n-k-2} + \&c.\right\}$$

$$B_k = b_k (1-\gamma^2)^{\frac{k}{2}} \left\{\gamma^{n-k} - \frac{(n-k)(n-k-1)}{2\cdot(2n-1)}\cdot\gamma^{n-k-2} + \&c.\right\}$$

where a_k b_k are arbitrary constants.

Now by Prop. v., if we put $1-2t=\gamma$, we have

$$P_n = c\left\{\gamma^n - \frac{n(n-1)}{2(2n-1)}\gamma^{n-2} + \frac{n\cdot(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\cdot\gamma^{n-4}\ \&c.\right\}$$

$$\therefore\ \frac{d^k P_n}{d\gamma^k} = c'\left\{\gamma^{n-k} - \frac{(n-k)(n-k-1)}{2(2n-1)}\cdot\gamma^{n-k-2} + \&c.\right\}$$

where c and c' are independent of γ.

$$\text{Hence,}\quad A_k = a_k \sin^k(\theta)\cdot\frac{d^k P_n}{d\gamma^k}$$

$$B_k = \beta_k \sin^k(\theta)\cdot\frac{d^k P_n}{d\gamma^k},$$

where a_k β_k represent arbitrary constants.

Hence the required function in its most general form, is

$$u = a_0 P_n + (a_1\cos\phi + \beta_1\sin\phi)\cdot\sin\theta\frac{dP_n}{d\gamma}$$

$$+ (a_2\cos 2\phi + \beta_2 \sin 2\phi)\sin^2\theta\cdot\frac{d^2P_n}{d\gamma^2} + \&c.$$

Prop. XII.

If Y_m, Z_n are any rational and entire functions which satisfy respectively the equations

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{dY_m}{d\gamma}\right\}+\frac{1}{1-\gamma^2}\cdot\frac{d^2Y_m}{d\phi^2}+m(m+1)\,.\,Y_m=0,$$

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{dZ_n}{d\gamma}\right\}+\frac{1}{1-\gamma^2}\cdot\frac{d^2Z_n}{d\phi^2}+n(n+1)\,Z_n=0,$$

it is required to find the value of $\int_\gamma\int_\phi Y_m Z_n$ between the limits $\gamma=-1$ and $\gamma=+1$, and from $\phi=0$ to $\phi=2\pi$.

Putting $\gamma=\cos\theta$, we have by the last proposition,

$$Y_m=a_0P_m+(\alpha_1\cos\phi+\beta_1\sin\phi)\,.\,\sin\theta\frac{dP_m}{d\gamma}$$
$$+(\alpha_2\cos2\phi+\beta_2\sin2\phi)\sin^2\theta\frac{d^2P_m}{d\gamma^2}\ \&c.,$$

and

$$Z_n=a_0P_n+(a_1\cos\phi+b_1\sin\phi)\,.\,\sin\theta\frac{dP_n}{d\gamma}$$
$$+(a_2\cos2\phi+b_2\sin2\phi)\sin^2\theta\,.\,\frac{d^2P_n}{d\gamma^2}\ \&c.;$$

where a_0, α_1, β_1, α_2, β_2, &c., and a_0, a_1, b_1, &c. are any arbitrary constants.

Multiply both these series together, and integrate the product with respect to ϕ, from $\phi=0$ to $\phi=2\pi$, observing that the integral of the product of the cosines or sines of unequal multiples of ϕ vanishes between the given limits; hence,

$$\int_\phi Y_m Z_n=2\pi\left\{a_0a_0P_mP_n+\tfrac{1}{2}(a_1\alpha_1+b_1\beta_1)\sin^2\theta\frac{dP_m}{d\gamma}\cdot\frac{dP_n}{d\gamma}\right.$$
$$+\tfrac{1}{2}(a_2\alpha_2+b_2\beta_2)\sin^4\theta\frac{d^2P_m}{d\gamma^2}\cdot\frac{d^2P_n}{d\gamma^2}\ \&c.$$

The general term of this series is evidently a constant quantity multiplied by

$$\sin^{2r}(\theta) \cdot \frac{d^r P_m}{d\gamma^r} \cdot \frac{d^r P_n}{d\gamma^r};$$

and if we make $\gamma = 1 - 2t$ and $t' = 1 - t$ as before, we have

$$\int_\gamma \sin^{2r}\theta \frac{d^r P_m}{d\gamma^r} \cdot \frac{d^r P_n}{d\gamma^r} = 2 \int_t (tt')^r \cdot \frac{d^r P_m}{dt^r} \cdot \frac{d^r P_n}{dt^r},$$

which by Prop. VII. Cor. 3. $= 0$, when m and n are unequal, but when $m = n$, its value is

$$\frac{2}{2n+1} \cdot (n+r)(n+r-1) \ldots\ldots (n-r+1).$$

Hence when m and n are unequal,

$$\int_\gamma \int_\phi Y_m Z_n = 0;$$

but when $m = n$,

$$\int_\gamma \int_\phi Y_n Z_n = \frac{4\pi}{2n+1} \{a_0 \alpha_0 + \tfrac{1}{2} \cdot n(n+1)(a_1 \alpha_1 + b_1 \beta_1)$$
$$+ \tfrac{1}{2}(n-1) \cdot n \cdot (n+1)(n+2)(a_2 \alpha_2 + b_2 \beta_2) + \&c.\}$$

COR. 1. If we represent the general value of Z_n by $F(\theta, \phi)$, and suppose Y_n to be the coefficient of h^n in the expansion of the quantity,

$$\{1 - 2h[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')] + h^2\}^{-\frac{1}{2}},$$

which by Prop. IX. evidently satisfies the equation in the enunciation of the present proposition, then shall

$$\int_\gamma \int_\phi Y_n Z_n = \frac{4\pi}{2n+1} F(\theta', \phi'),$$

that is, it is the same as Z_n when θ and ϕ are changed into θ' and ϕ', and then multiplied by $\frac{4\pi}{2n+1}$.

For if in the expansion of Y_n given in Prop. X., we put for $F_k(\gamma)$ its value found in the succeeding proposition, we have

$$Y_n = P_n P'_n + \frac{\sin\theta\sin\theta'}{n(n+1)} \cdot \frac{dP_n}{d\gamma} \cdot \frac{dP'_n}{d\gamma'} \cdot 2\cos(\phi - \phi')$$
$$+ \frac{\sin^2\theta\sin^2\theta'}{(n-1)n(n+1)(n+2)} \cdot \frac{d^2P_n}{d\gamma^2} \cdot \frac{d^2P'_n}{d\gamma^2} \cdot 2\cos 2(\phi - \phi') \text{ \&c.}$$

where $\gamma' = \cos\theta'$ and P'_n is the same function of γ' that P_n is of γ; then comparing this value with the form assumed for Y_n in the present proposition,

we have $a_0 = 1$,

$$a_1 = \frac{2\sin\theta'\cos\phi'}{n(n+1)} \cdot \frac{dP'_n}{d\gamma'},$$

$$\beta_1 = \frac{2\sin\theta'\sin\phi'}{n(n+1)} \cdot \frac{dP'_n}{d\gamma'},$$

&c. = &c.

substituting these values, we obtain

$$\int_\gamma \int_\phi Y_n Z_n = \frac{4\pi}{2n+1} \left\{a_0 + (a_1\cos\phi' + b_1\sin\phi')\sin\theta' \frac{dP'_n}{d\gamma}\right.$$
$$\left. + (a_2\cos 2\phi' + b_2\sin 2\phi')\sin^2\theta' \frac{d^2P'_n}{d\gamma^2} + \text{\&c.}\right\},$$

which is evidently what the general value of Z_n becomes, when θ and ϕ are respectively changed into θ' and ϕ'; and the result multiplied by $\frac{4\pi}{2n+1}$.

COR. 2. A process similar to that employed in Prop. IV. will equally apply to the expansion of a given function y, in terms of functions of the same nature as Z_n; that is, such as satisfy the equation

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{dZ_n}{d\gamma}\right\} + \frac{1}{1-\gamma^2} \cdot \frac{d^2Z_n}{d\phi^2} + n(n+1)Z_n = 0.$$

CHAPTER II.

STATEMENT OF THE ORDINARY ELECTRICAL PHÆNOMENA.

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(1). *Production of Electricity.*

When a piece of amber, sealing-wax, &c. or a plate of glass is rubbed briskly with a woollen cloth, and then held near small pieces of paper or other light substances, some are observed to fly towards the rubbed body, and attaching themselves to it for a short time, to fly off again, while others having arrived within a short distance of that body, suddenly fly back without touching it. These phænomena are called electrical from the Greek 'ελεκτρον' *amber*, and in this case the body which exerts such attractions and repulsions is said to be *electrised by Friction*.

Electricity is also produced when bodies undergo a change of state, as when melted sulphur is poured into a metallic pot, supported on glass legs, and allowed to solidify by cooling; in this state both the sulphur and the vessel exhibit signs of electricity.

Pressure as in the case of the topaz, and heat as instanced in tourmaline develop electricity.

The organization also of certain animals, as the torpedo, produces effects similar to the electrical, particularly in the shock received when touched by the hand.

(2). *Communication of Electricity.*

Let a cylinder of glass, the axis of which rests on fixed supporters, be made to revolve rapidly by means of a handle attached to the axis.

During the rotation, let a silk cushion stuffed with hair be pressed in a fixed position against the cylinder, by means of a strong and elastic lamina of iron about an inch broad, to which it is attached, and which at its lower extremity communicates with the ground; the cylinder will thus be rendered electrical by friction.

Let a hollow metallic cylinder, of which the length is about three or four times the breadth, be placed on a dry glass supporter, and then let one end of it armed with sharp points, be brought near the revolving cylinder.

If the room, in which the experiment is made, be darkened, streams of light will be observed at short intervals, rushing from the electrified glass cylinder to the sharp points, accompanied with a crackling noise.

After this process is continued for a few minutes, let the metallic cylinder be removed, holding it by the glass supporter; it will then be found to possess the electrical properties, in attracting and repelling light bodies, and in communicating a slight shock when touched by the hand.

Such is the principle on which the common electrical machine is constructed, where the glass cylinder rendered electrical by friction has evidently communicated electricity to the metallic cylinder placed near it; and it may be observed that another metallic body with a glass handle, would by merely touching the former, acquire similar properties.

(3). *Electrical Influence.*

Suppose that electricity is produced, as in the preceding article, and communicated to a metallic globe resting on a glass supporter. Let the globe be then brought near a thin metallic cylinder resting also on a glass supporter, the latter will then be found to possess electrical properties, which may be exhibited by bringing light bodies near its surface, or more simply by suspending at different points of the cylinder, pairs

of pith balls connected by fine flaxen threads passing over the cylinder; the pair of balls which are suspended at the extremity of the cylinder nearest the globe, will recede from each other through a large angle; those near the middle of the cylinder will scarcely have any divergence, but those at the middle and beyond, diverge more and more, the nearer we approach the second extremity of the cylinder.

Let now the cylinder be removed from the vicinity of the globe, the former will instantly lose all its electrical properties, and the balls will return to their natural positions: thus the mere presence of an electrised body induces an electrical state in adjacent bodies resting on glass supporters.

(4). *Conductors and Non-conductors.*

The metallic bodies, which we have supposed placed on glass supporters, are said to be *insulated;* for if instead of glass a metallic supporter were used, with its lower extremity in contact with the ground, the effect of this communication would be to deprive the electrised bodies of the properties they had acquired, and to restore them to their natural state.

Let now a metallic rod be insulated and electrised, and let a series of insulated metallic rods in their natural state, be placed in successive contact with each other; touch then the first of the series with the electrised rod, they will all become electrical almost instantaneously; but if one part of the series had been a stick of sealing-wax or a glass rod, this part would neither acquire electricity itself, nor suffer the metallic rods beyond it to acquire any, but would completely cut off the communication with the electrised rods; it is therefore said to be a *non-conductor* of electricity, and such substances are therefore used for insulators; on the other hand, the metallic bodies by which the communication may be prolonged to any extent, are named *conductors.*

Metallic substances and liquids are mostly conductors; gums, vitrefactions and dry gases non-conductors; there is however but little probability that there exists any substance

which is either a perfect conductor, or an absolute non-conductor; thus the glass supporters in the preceding experiments, always exhibit some small signs of an electrical state, at that extremity which is in contact with the electrised metallic body.

(5). *Positive and Negative Electricities.*

Let an insulated metallic globe (A), and a pith ball (a), of very small dimensions compared with the globe, and suspended by a silk string, be both electrised by communication with the machine described in *Art.* (2). In like manner let another globe (B) and another pith ball (b) be electrised by means of a machine, which differs from the former in having the revolving cylinder composed of some resinous substance instead of glass.

The following phænomena will then occur:

When the ball (a) is brought near the globe (A), it will be repelled with great energy; and in like manner will (b) be repelled when brought near (B).

On the other hand, when the ball (b) is brought near the globe (A) or the ball (a) near the globe (B), the balls will no longer be repelled, but attracted towards the globes.

The ball (a) and globe (A), in electrising which the glass revolving cylinder was employed, are said to be charged with *vitreous* or *positive* electricity, and the ball (b) and globe (B) with *resinous* or *negative* electricity: the preceding phænomena may then be announced by saying that electricities of a *like* kind repel, and of *unlike* kinds attract.

If instead of the globe (A) a pith ball (a') of equal dimensions with (a) and similarly electrised, had been used, then the two balls would mutually repel; the reason that (A) is apparently not repelled by (a) is merely that its mass is so much greater; for if it were also suspended by a fine string, it would in reality begin to recede from (a) with a velocity which would

be to the initial velocity of (*a*) as the mass of (*a*) to that of (*A*), in consequence of the necessary equality between action and reaction; but the action of gravitation tending to restore it to its original position, the angle through which it moves in the case above considered would be quite insensible.

When the ball (*a*) is observed to recede from the globe (*A*), we are not to infer that the electricity in (*a*) being repelled by that in (*A*) draws along with it the matter of the ball (*a*) by cohesion to its particles, because the conductibility of the ball renders the supposition of such cohesion very improbable; and it is at the same time obvious, that the phænomenon would be produced merely by rendering the pressure of the electricity against the surrounding air unequal, for the air being a non-conductor will not suffer it to escape, and reacting unequally on different points of the surface of the ball, would produce the observed motions.

Note. No very general law has been observed with respect to the nature of the substances by the friction of which either kind of electricity may be produced; it is however universally true that the rubbing and the rubbed substances always acquire opposite electricities, from whence the inconvenience of the names, *vitreous* and *resinous*, as applied to electricity, is apparent; for if two plates of glass or two pieces of resin be rubbed together, one of the glass plates will acquire resinous, and one of the pieces of resin, vitreous electricity.

(6). *Capacity for Electricity.*

When plates of different substances, but of the same form and dimensions, are charged by means of the common electrical machine, the number of turns made by the revolving cylinder, and consequently the quantity of electricity communicated before its escape, which is indicated by the electric spark, varies according to the material of which the subtance is composed, and being found greatest in non-conductors, they are therefore said to have the greatest capacity for electricity.

The mode of electrising a non-conductor, as for instance a plate of glass, is the following; place it on a conducting surface communicating with the ground, and also put a metallic plate in contact with the upper surface; having charged this plate with electricity, remove it by means of insulating handles, the glass will then be found strongly charged with electricity.

(7). *Electrical Instruments.*

A variety of instruments have been constructed for different purposes connected with experiments on electricity; as our object is to describe merely the most simple and useful, we shall take only the Electroscope, Proof-plane, Torsion-balance, and Leyden Jar.

Electroscope. A pair of small and extremely light balls made of elder-pith and connected by a fine silk thread taken from the cacoon, form when well dried, a simple and useful instrument for indicating the presence of electricity; when the string is held by the middle so that the balls hang together and are made to touch a body slightly electrised, they instantly both recede from that body, and diverge from each other; if they are taken enclosed in a glass cage to the upper strata of the atmosphere in fine weather, their divergence continually increasing shews it to be in an electrical state, and even at the surface the same phænomenon frequently occurs, when a cloud charged with electricity moves over-head at a small altitude.

Proof-plane. This useful instrument consists of a very small disc of gilt paper attached to the extremity of a good insulator, which generally is a filament of gum-lac; it is applied in measuring the intensity of electricity at the different points of the surface of a body, for by contact with any point, it will take when very small, an electrical charge proportional to the quantity of electricity at that point; immediately on being removed it should be applied to a delicate electrometer, such as that which will be next described.

A natural application has been made of this instrument, to shew experimentally that when a conducting body is electrised, none of the free electricity remains in the interior; but the same may be also proved by enveloping the body with a conducting cover, as paper, which if composed of two parts which meet so as exactly to cover the body, may be put on and taken off by means of insulating handles; if the substance be electrised and the envelop then put over it, the paper when removed will be found charged with electricity, while the body will be completely deprived of it, thus demonstrating that all the free electricity resides on the surfaces of bodies.

Torsion Balance. Suppose a needle of gum-lac is suspended horizontally by an extremely fine silk thread perfectly unravelled, if we move the needle from its natural position in a horizontal plane, the torsion which the thread undergoes will tend to restore it to its primitive place, with a force proportional to the angle of torsion, which is here the same as the angle through which the needle is made to deviate; this angle may be observed by enclosing the apparatus in a glass cylinder round which there is a graduated band in the same horizontal plane as the needle.

Another torsion may be communicated to the thread from its upper extremity, to effect which, a graduated brass plate to the center of which the thread is attached, moves with friction on the top of a smaller glass cylinder placed on the former, and a stationary index will shew the angle through which it has turned.

When both torsions are in opposite directions, the whole torsion is their sum, and in the same directions, their difference.

To instance the use of this instrument, suppose a pith ball (A) attached to the end of a needle of gum-lac, while another (B) is attached to a *fixed* insulating supporter, so as to be in contact with the former; when the balls are electrised, the first will be repelled through an angle (ω), and if we make the

length of the needle a unit, the distance or chord $AB = 2\sin\frac{\omega}{2}$, and the repulsive force, which is some function of this distance, may be represented by $f(\omega)$, while the force of torsion, which balances the repulsion and is as the angle of torsion may be represented by $m\omega$, m depending on the material of the thread only; hence we must have

$$m\omega = f(\omega)\cos\frac{\omega}{2}.$$

If now the plate at the top be made to revolve, so as to bring (A) nearer to the fixed ball (B), another position of equilibrium will be formed; and representing the total torsion by t and the *arc* AB in this position of the ball (A) by θ, we have

$$mt = f(\theta)\cos\frac{\theta}{2}.$$

Hence, $$\frac{t}{\omega} = \frac{f(\theta)\cos\frac{\theta}{2}}{f(\omega)\cos\frac{\omega}{2}}$$

similarly, for a third position of A when the arc $AB = \theta'$ and the torsion $= t'$, we have

$$\frac{t'}{\omega} = \frac{f(\theta')\cos\frac{\theta'}{2}}{f(\omega)\cos\frac{\omega}{2}},$$

and so on for any number of positions.

Now since all the quantities t, t', &c. and ω, θ, θ', &c. are measured by the construction of the instrument, we can try if any assumed form of $f(\theta)$ will verify the above equations, and from Coulomb's experiments it appears that when $f(\theta) = \frac{a}{\operatorname{vers}\theta}$, these equations will be satisfied with remarkable

exactness; but the versed sine of the arc AB is as the square of the chord, hence the repulsion of (B) on the ball (A) must be proportional to the inverse square of the distance.

If instead of the ball (B) the proof-plane above described be used, and the arc AB through which (A) recedes be put equal to ω, the corresponding intensity of the electricity on the proof-plane being i, and on another trial these quantities are ω' and i'; then assuming the law of electric action to vary as the inverse square of the distance, we have

$$\frac{i'}{i} = \frac{\omega'}{\omega} \cdot \frac{\text{vers.}\,\omega'}{\text{vers.}\,\omega},$$

thus by combining the proof-plane and torsion-balance, we may form a very exact electrometer.

Leyden Jar. Let a glass bottle be lined inside and outside with tin-foil, except a small portion towards the top; let the inside be electrised *positively*, by means of a brass chain communicating with the prime conductor of an electrical machine, and also with the interior foil at the bottom of the jar.

The positive electricity in the interior will attract and therefore detain the negative electricity of the exterior foil, but it will repel the positive which will thus escape into the ground, on which we may suppose the jar placed; the exterior foil will therefore be *negatively* electrised.

If now a communication be made between the internal and external electricities, by placing for instance one hand on the outside foil and the other on the chain conducting to the inside, a smart shock will be immediately felt; such is the principle on which the Leyden jar is constructed.

A series of such bottles communicating with each other, and with the prime conductor of an electrical machine, and placed in a wooden box lined with tin-foil, constitute what is called an electrical battery.

(8). *Gradual dissipation of Electricity.*

When a body is insulated and electrised, the intensity of the electricity rapidly diminishes (particularly when the atmosphere contains much moisture) so as ultimately to become quite insensible.

The causes of this loss of intensity are:

First. The imperfect insulation of the body; for as was stated in Art. 4, there is no known substance which may be regarded as an absolute non-conductor.

Secondly. The surrounding air, the particles of which become electrised by contact with the body, and then are repelled by it, while other particles coming successively in contact with it, carry off in like manner additional portions of electricity.

Thirdly. The humidity of the air which deposits on the insulator small globules of water, and thus forms a conducting chain reaching from the body to the ground: this which is the principal cause depends on the hygrometric state of the atmosphere.

When the latter cause is avoided, by making the experiments in fine weather, the loss of intensity in a short interval of time due to the first two causes, is found to be proportional to the actual intensity; hence if i be the intensity at any time t, and m a constant quantity depending on the nature of the electrised body, we have

$$-\frac{di}{dt} = mi;$$

$$\text{therefore,} \quad i = c \,.\, \epsilon^{-mt};$$

where c represents the initial intensity: hence in equal successive portions of time the intensity diminishes in a geometrical progression.

(9). *Effects of Electricity.*

When a conducting body is charged with electricity of either kind, a pressure due to the mutual repulsion of the electrical particles will be exercised against the air which is a non-conductor; this pressure is generally different at different points of the surface of the conductor, but at a given point it will depend on the whole charge, and will increase with it. If by adding to the charge we increase the pressure until it is sufficient to overcome the resistance offered by the air, it will escape from the conducting body at that point in which it has the greatest intensity. This tendency to escape, may be increased by placing another conducting body near the former; the natural electricities of this conductor being decomposed by *influence*, that which is unlike the electricity of the charge becomes collected on that part of the second conductor which is near the first, and by its attraction evidently increases the tendency of the electricity in the first conductor, to escape.

The escape of the electricity is indicated by a spark or stream of light visible between the conductors, caused by the great pressure exercised against the air during the passage of the electricity, the velocity of which is immensely great; but if the electricity be discharged through an exceedingly fine wire, the latter will be rendered red-hot, indicating the agitation of the molecules of which it is composed.

A similar experiment has been made with long metallic bars, and the result has been a sensible and permanent alteration of magnitude.

In like manner, a moderate electrical charge passed through fluids contained in narrow glass tubes, produces so rapid an expansion of the fluids that the tubes are often burst.

Water may be formed by passing the electric spark in a jar containing hydrogen gas and common air, or oxygen.

If a metallic rod, terminated in a fine point, and covered with an isolating substance, except at that point which is in contact with water be electrised, the latter will be decomposed into its constituent gases.

Many other chemical effects are produced by electricity, but it will be sufficient here to state that oxygen generally combines with bodies positively electrised, and the combination is favoured by raising the temperature of the body.

Vegetable as well as animal life may be destroyed by the electric shock, but in the latter instance, more moderate shocks have been applied with doubtful success in such diseases as nervous contractions, deafness, rheumatism, &c.

The colour of the electrical spark depends partly on the substance to which electricity is transmitted, partly on the medium through which it passes; the more distant we place that substance, or the more the medium is rarified, the whiter will be the colour of the spark. When examined by viewing it through a prism, the effects are similar to those of common light.

(10). *Diffusion of Electricity.*

Both the electricities are contained in every known substance, and the quantities of each kind susceptible of development, when the body is in its natural state, are generally equal.

If the body is a conductor, and is brought near another body positively or negatively electrised, its electricities will be partially separated and in equal quantities.

When chrystalline bodies exhibit electricity in consequence of a change of temperature, the electricity at one angle or pole is positive, and negative at the opposite one.

The atmosphere, when pure, is in a state of negative electricity, the more intense as we ascend to higher regions, where

its density is less; but when rendered impure by breathing, in a close room, it is found to be positively electrised.

When a solid body undergoes fracture, or is torn asunder, the separated surfaces are found oppositely electrised.

In the formation of gases and the condensation of vapours, electricity is developed; thus snow and hail, immediately after falling, are found negatively electrised.

Thunder and lightning are effects of electricity, and the heavy showers of rain which frequently accompany this phænomenon, may be conceived to be produced by the electric discharge, bringing into combination, the oxygen and hydrogen gases at that time in the atmosphere, in the same manner that those gases when contained in a glass vessel, are combined by passing through them the electrical spark.

Many other meteoric phænomena, as the Aurora Borealis, &c. seem referable to the same cause; and some experiments not sufficiently confirmed, would seem to indicate that even solar light possessed some electrical qualities.

CHAPTER III.

ON ELECTRICITY IN ITS LATENT STATE.

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(11). *Definition of Latent Electricity.*

WHEN a conducting body is taken in its natural state and applied to the most delicate electrometer, it will generally evince no signs of electricity; but when brought near an electrised body its opposite ends will then be found to be electrised, the one positively, the other negatively; it will however be instantaneously restored to its natural state, by removing the influencing body. (Art. 3). The electricities which have thus subsided into a state of neutralization, are here called latent; and the mathematical character of latent electricity, is, that the total action of the system on an external point, is zero.

(12). *Mathematical Signs of the Electrical Forces.*

Conceive a fixed electrical particle P, of either kind of electricity, to act on a *positive* particle p which is free to move, let the distance Pp be represented by x, P being the origin, and the right line joining the particles, the axis of x; let ρ represent the mass of P, and $f(x)$ the law according to which the force varies at different distances. Then t representing the time, the equation for the motion of p is

$$\rho f(x) = \frac{d^2x}{dt^2},$$

$$\text{or,}\ \rho f(x) = -\frac{d^2x}{dt^2},$$

according as the action is repulsive or attractive, that is, according as the electrical particle P is positive or negative;

this ambiguity is removed by giving ρ the sign expressed by the name of that kind of electricity, of which it represents the quantitative accumulation in P; for the second equation would then be

$$-\rho f(x) = -\frac{d^2 x}{dt^2},$$

which is the same as the first.

With this convention, the *nature* of the action between two electrical particles will be indicated by the sign of the product of their masses, the sign + expressing repulsion and − attraction.

It is moreover evident that the formula

$$\int_x \int_y \int_z \rho,$$

extended through the whole of a given system, will not express the whole quantity of electricity in that system, but merely the excess of the quantity of positive, above that of negative electricity.

The statical effect of an electrical system, which, estimated in any direction, is the difference between the effects of all the attracting and of all the repelling particles, will by this notation be expressed by merely one integral instead of the difference between two sums.

(13). *To find the law of force, tending to or from each electrical particle.*

When a sphere is electrised by communication, the whole quantity of developed electricity resides on the surface; (Art. 7. *Proof-plane*) and it is evident that its distribution there will be uniform; moreover, it is necessary for the equilibrium of the latent electricity in the interior of the sphere, that the total action of this external stratum on any internal point, must be zero, the system of latent electricity being of itself in equilibrium.

Take therefore any point P within the sphere, at a distance h from the centre O, make the centre the origin, and the right line OP joining the given point, and centre the axis of x, and let r be the radius of the sphere.

Let a section of the spherical surface be made by a plane drawn perpendicular to the axis of x, at a distance x from the origin, and let f be the distance of any point in this section, from the assumed point P.

If we take another section made by a plane at a very small distance $\delta(x)$, the total attraction of the annulus between both sections on the given point P, will be directed in the line OP, and be represented by $2\pi r \cdot \phi(f) \cdot \dfrac{h-x}{f} \cdot \delta x$, where $\phi(f)$ expresses the required law of force at different distances.

But since the whole action on P is zero, we have

$$\int_x \phi(f) \cdot \frac{h-x}{f} = 0, \text{ from } x = -1 \text{ to } x = +1.$$

$$\text{Now } \quad f^2 = r^2 + h^2 - 2hx;$$

$$\text{hence } \frac{df}{dh} = \frac{h-x}{f}, \text{ and } \frac{df}{dx} = -\frac{h}{f};$$

$$\text{therefore, } \int_x \phi(f) \cdot \frac{df}{dh} = 0, \text{ between the above limits.}$$

Let $\phi(f)$ be the differential coefficient of $\phi_1(f)$, taken with respect to f as variable.

$$\text{Hence } \frac{d}{dh} \int_x \phi_1(f) = 0;$$

$$\text{or, } \frac{d}{dh} \int_f \phi_1(f) \cdot \frac{f}{h} = 0, \text{ since } \frac{dx}{df} = -\frac{f}{h}.$$

Again, let $f\phi_1(f)$ be the differential coefficient of $\psi(f)$, then observing that the limits of f are $r-h$ and $r+h$, we get

$$\frac{d}{dh}\left\{\frac{1}{h}\cdot\psi(r+h)-\frac{1}{h}\psi(r-h)\right\}=0;$$

and expanding the functions by Taylor's theorem,

$$2\frac{d}{dh}\left\{\frac{d\psi(r)}{dr}+\frac{h^2}{1.2.3}\cdot\frac{d^3\psi(r)}{dr^3}+\&c.\right\}=0;$$

hence, $\dfrac{2h}{1.2.3}\cdot\dfrac{d^3\psi(r)}{dr^3}+\dfrac{4h^3}{1.2.3.4.5}\cdot\dfrac{d^5\psi(r)}{dr^5}+\&c.=0,$

this evidently requires each term to be separately equal to zero;

$$\text{hence, } \frac{d^3\psi(r)}{dr^3}=0, \text{ or, } \frac{d^3\psi(f)}{df^3}=0.$$

Now since $\dfrac{d\psi(f)}{df}=f\phi_1(f)$;

$$\text{therefore, } \frac{d^2\psi(f)}{df^2}=\phi_1(f)+f\frac{d\cdot\phi_1(f)}{df}$$

$$\frac{d^3\psi(f)}{df^3}=2\frac{d\phi_1(f)}{df}+f\frac{d^2\phi_1(f)}{df^2}$$

$$=2\phi(f)+f\frac{d\phi(f)}{df};$$

substituting this value, we have

$$\frac{d\phi(f)}{df}=-2\cdot\frac{\phi(f)}{f};$$

and integrating $\log\phi(f)=\log(A)-2\log(f)$, A representing a constant quantity,

$$\text{hence } \phi(f)=\frac{A}{f^2},$$

that is, the force varies inversely as the square of the distance.

(14). *If* a *be the distance of any point* P *from a fixed point* O, *and* V *represent the sum of all the molecules of an electrical system when divided by their respective distances from* P, *the action of the whole system on* P *in the direction* OP *will be represented by* $\frac{dV}{da}$.

Make the fixed point O the origin of co-ordinates, and OP the axis of x; let f be the distance of any molecule from P, δm the bulk, and ρ the density of this molecule, and x, y, z its co-ordinates; then since

$$f = \{(a-x)^2 + y^2 + z^2\}^{\frac{1}{2}},$$

$$\text{therefore, } \frac{d\left(\frac{1}{f}\right)}{da} = \frac{a-x}{\{(a-x)^2 + y^2 + z^2\}^{\frac{3}{2}}}$$

$$= \frac{a-x}{f^3}$$

$$= \frac{1}{f^2} \cdot \frac{a-x}{f}.$$

Now the force exerted by this particle on the point P in the direction of f, is represented by $\frac{\rho\delta . m}{f^2}$, which being resolved in the direction of the right line OP, gives

$$\frac{\rho\delta m}{f^2} \cdot \frac{a-x}{f} = \frac{d}{da}\left(\frac{\rho\delta m}{f}\right),$$

the density ρ being a function of the co-ordinates x, y, z only.

But since V is the sum of all the quantities $\frac{\rho\delta m}{f}$ extended throughout the entire system, therefore, the total action on P in the direction OP, is evidently represented by $\frac{dV}{da}$.

(15). *When electricity is latent in any body, the quantity of positive electricity contained in the body, is equal to that of negative.*

Let r be the distance of any external point P, from a fixed point O taken within the body.

Make O the origin of co-ordinates, and let θ be the angle which OP makes with the axis of x, and ϕ the inclination of the plane in which θ lies to the plane of $x\,y$.

Again, let ρ be the electrical density at any point p, within the body of which the polar co-ordinates measured in like manner, are r', θ', ϕ'.

Then since the cosine of the angle $p\,OP$ included between r and r' is evidently $\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, if we represent the distance Pp by f, we have

$$\frac{1}{f} = \{r'^2 - 2rr'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')] + r^2\}^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left\{Q_0 + Q_1\frac{r'}{r} + Q_2 \cdot \frac{r'^2}{r^2} + \&c.\right\},$$

adopting here the notation used in Prop. VIII.

Also since the mass of the electrical particle p is represented by $\rho r'^2 \sin\theta' \,.\, \delta r'\,\delta\theta'\,\delta\phi'$, it follows that the whole action on P in the direction OP, is, by Art. (14),

$$\frac{d}{dr}\left\{\int_{r'}\int_{\theta'}\int_{\phi'}\rho r'^2\sin\theta'\left(\frac{Q_0}{r} + Q_1\frac{r'}{r^2} + Q_2\frac{r'^2}{r^3} + \&c.\right)\right\},$$

which must be equal to zero since the electricity is latent;

$$\text{hence,}\quad \frac{1}{r}\left\{\int_{r'}\int_{\theta'}\int_{\phi'}\rho r'^2\sin\theta'\left(Q_0 + Q_1\frac{r'}{r} + Q_2\frac{r'^2}{r^2} + \&c.\right)\right\} = c,$$

c being a quantity independent of r, and since P is only limited to be an external point, we may put $r = \infty$, which gives $c = 0$.

Suppose now the integrations actually performed, putting

$$A = \int_{r'} \int_{\theta'} \int_{\phi'} Q_0 \rho r'^2 \sin\theta',$$

$$B = \int_{r'} \int_{\theta'} \int_{\phi'} Q_1 \rho r'^3 \sin\theta',$$

&c.

and the integrals being extended throughout the entire system, we obtain thus,

$$\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \&c. = 0,$$

which requires $A = 0$, $B = 0$, $C = 0$, &c.

Now since $Q_0 = 1$, it is evident that A is the excess of the quantity of positive above that of negative electricity in the entire system, (Art. 12); hence, when the electricity is latent, those quantities are necessarily equal.

Note. Though the quantities of both electricities must be equal in the latent state, this condition alone is not sufficient to produce that state, since there are besides an infinite number of equations, $B = 0$, $C = 0$, &c. also to be satisfied.

(16). *To find the law of the distribution of latent electricity in an indefinitely thin rod of any material.*

Take the length of the rod as a unit, and let t be the distance of any point (p) in the rod, from one extremity, and h the distance of a point (P) in the direction of the rod produced from the same extremity; and therefore, $h - t$ will be the mutual distance of both points.

Let ρ be the electrical density or accumulation at the point p, then since the electricity is latent, the total action on the external point P, is zero; and consequently if each electrical particle be divided by its distance from P, the sum of the quotients is constant (Art. 14), if moreover we suppose P at an infinite distance, it is evident that this constant is zero.

Hence, $\int_t \frac{\rho}{h-t} = 0$, from $t = 0$ to $t = 1$;

$$\text{that is, } \int_t \rho \left(\frac{1}{h} + \frac{t}{h^2} + \frac{t^2}{h^3} + \&c.\right) = 0,$$

which requires that we have, separately,

$$\int_t \rho = 0, \ \int_t \rho t = 0, \ \int_t \rho t^2 = 0, \ \&c. \ldots\ldots (a).$$

The number of equations which it is necessary to satisfy, is therefore infinite; we may first suppose that they are only n in number, and then put $n = \infty$.

If we make $1 - t = t'$, as in Prop. I., and refer to the reasoning of that proposition, it will be obvious that $t^n t'^n$ is a factor of the n^{th} integral of ρ, with respect to t; all the successive integrals commencing when $t = 0$; if the other factor which remains arbitrary and can depend only on the physical constitution of the rod, be called T, then we have

$$\rho = \frac{d^n}{dt^n} \cdot (T t^n t'^n),$$

and when we make n infinite, this quantity expresses the law of the electrical distribution.

It is evident that with this arrangement, the electricity will be latent with respect to all external points, whether in the direction of the rod produced or not; for the reciprocal of the distance of the point p from any such external point, may be expanded in a series of the form

$$A_0 + A_1 t + A_2 t^2 + \&c.;$$

and when we multiply by ρ and integrate, the result is zero by the equations (a), which ρ has been made to satisfy; hence, the total action must also be nothing.

Corollary. Suppose the above value of ρ expanded in a series of the form

$$a_0 P_0 + a_1 P_1 + a_2 P_2 + \ldots\ldots a_{n-1} P_{n-1} + a_n P_n + \&c. \text{ by Prop. VI.}$$

then it is evident by the equation (a), that when multiplied by P_m, and then integrated from $t=0$ to $t=1$, the integral must vanish when $m<n$, since the terms of which P_m is formed, consist of powers of t less than the n^{th}; but by Prop. IV. the same integral is equal to $\frac{a_m}{2m+1}$, hence $a_m=0$, and consequently we have

$$\rho = a_n P_n + a_{n+1} P_{n+1} + a_{n+2} P_{n+2} + \&c. ;$$

that is, the electrical distribution in a rod of any material, may be conceived to be formed by the superposition of several systems of which the general *type* is P_n, when n is indefinitely great.

(17). *When electricity is latent in a straight rod, there are an indefinitely great number of points at which the electricity is neutral.*

The equation $Tt^n t'^n = 0$, when solved with respect to t, has n real roots each $=0$, and n real roots each $=1$; and it may have more real roots due to the real factors of T.

Hence the limiting equation

$$\frac{d}{dt}(Tt^n t'^n) = 0$$

has $(n-1)$ roots each $=0$, $(n-1)$ roots each $=1$, and at least one real root between 0 and 1.

Similarly the equation

$$\frac{d^2}{dt^2}(Tt^n t'^n) = 0$$

has $(n-2)$ roots each $=0$, $(n-2)$ roots each $=1$, and at least two real roots between 0 and 1.

Continuing this process, it follows that the equation

$$\frac{d^n}{dt^n}(Tt^n t'^n) = 0$$

has at least n real roots between 0 and 1.

Now this last equation determines the positions of the *neutral* points, for at such points we must have $\rho = 0$; and since n is an indefinitely great number, and the length of the whole rod being comprehended between $t = 0$ and $t = 1$, it follows that there are an indefinitely great number of neutral points ranged along the rod.

(18). *To trace the distribution of latent electricity represented by the function* P_n, n *being indefinitely great.*

The equation $P_n = 0$, which is of n dimensions with respect to t, has all its roots real and lying between the limits 0 and 1; (Art. 17.) let them be represented by $t_1, t_2, t_3 \ldots t_n$, where t_1 is the least, and the remaining roots are arranged in the order of their magnitudes.

The points at which the electrical accumulation is a maximum, are defined by the limiting equation $\frac{dP_n}{dt} = 0$; the roots of which lie between the roots of the equation $P_n = 0$. Let $\tau_1, \tau_2, \tau_3 \ldots \tau_{n-1}$ represent the values of t at those points.

The order in which all the preceding quantities stand with respect to magnitude, is evidently the following:

$$t_1, \tau_1, t_2, \tau_2, t_3 \ldots \tau_{n-1}, t_n.$$

Conceive now the electricity positive at one extremity of the rod where $t = 0$, the corresponding value of the electrical accumulation P_n is then represented by $+1$, from which point it diminishes so as to vanish when $t = t_1$, after which the electricity becomes negative, and increases in intensity until $t = \tau_1$; it then diminishes and vanishes a second time, when $t = t_2$; the electricity after this becomes again positive, and so on; it is therefore on the whole disposed in n successive portions, containing alternately the positive and negative electricities: and since P_n is the coefficient of h^n in the expansion of

$$\{1 - 2h(1 - 2t) + h^2\}^{-\frac{1}{2}}, \text{ by Prop. II.,}$$

and therefore when $t = 1$, is the coefficient of h^n in $(1 + h)^{-1}$, it follows that the electrical accumulation at the second

extremity of the rod, is the same in quantity as at the first, but of the same or of a different name, according as n is even or odd.

If we denote by P'_n the electrical accumulation when we put for t its supplementary value $1-t$ or t', then we have by Prop. I.,

$$P_n = \frac{1}{1.2.3\ldots n} \cdot \frac{d^n (tt')^n}{dt^n}$$

$$P'_n = \frac{1}{1.2.3\ldots n} \cdot \frac{d^n (t't)^n}{dt'^n};$$

$$\text{hence, } P'_n = (-1)^n . P_n,$$

that is, the electrical accumulation is the same in quantity at any two points equi-distant from the middle of the rod, but is of the same or of different names according as n is even or odd.

Again, if we put $t = t' = \frac{1}{2}$ in the second expansion of Prop. V. we get

$$P_n = \frac{1}{2^n} \left\{ 1 - \binom{n}{1}^2 + \left\{ \frac{n.(n-1)}{1.2} \right\}^2 - \left\{ \frac{n(n-1)(n-2)}{1.2.3} \right\}^2 + \&c \right\}.$$

Now the quantity between the brackets is the part which does not contain h in the product,

$$\left\{ 1 - \frac{n}{1} . h + \frac{n(n-1)}{1.2} . h^2 - \&c \right\} \times \left\{ 1 + \frac{n}{1} . \frac{1}{h} + \frac{n.(n-1)}{1.2} . \frac{1}{h^2} + \&c \right\},$$

$$\text{that is, in } \frac{(1-h)^n (1+h)^n}{h^n};$$

it is therefore equal to the coefficient of h^n in $(1-h^2)^n$, and consequently is zero when n is odd, but when n is even, its value is

$$\frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}};$$

P_n is therefore also zero when n is odd, but when n is even, its value is

$$\frac{1}{2^n} \cdot \frac{n(n-1)(n-2)\ldots 1}{\left(1 \cdot 2 \cdot 3 \ldots \frac{n}{2}\right)^2} = \frac{1 \cdot 2 \cdot 3 \ldots n}{(2 \cdot 4 \cdot 6 \ldots n)^2}$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots n};$$

$$\text{hence, } \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (n-1)(n-1)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \ldots (n-2) \cdot n} = n \cdot P_n^2,$$

and making n indefinitely great, we get

$$P_n = \pm \sqrt{\left(\frac{2}{\pi n}\right)},$$

where π is the semicircumference of a circle of which the radius is unity: the electrical accumulation at the middle of the rod is therefore zero when n is odd, but when n is even, it is to the accumulation at the first extremity as $\sqrt{\left(\frac{2}{\pi n}\right)}$; 1; and it is positive or negative, according as n is of the form $4m$ or $4m + 2$, m being an integer.

Though the total quantity of positive electricity is equal to that of negative, (Art. 15.) and the whole is arranged in n successive cumuli of alternately positive and negative electricities, these cumuli are not to be regarded as equal to each other, but the first positive cumulus is equal and of a contrary name to that portion of the second which is terminated at the point of maximum intensity, the remaining portion of the second is equal and of a contrary name to that portion of the third, which is terminated at the point of its maximum intensity, and so on; for by Prop. VII. Cor. 1, $\int_t P_n$ vanishes from $t = 0$ to $t = \tau_1$, from $t = \tau_1$ to $t = \tau_2$, and generally from $t = \tau_m$ to $t = \tau_{m+1}$.

To determine the points of maximum intensity along the rod, suppose that we take its first extremity as origin, and

construct a curve in which the abscissa x is taken equal to t, and the ordinate $y = P_n$, this curve will cut the axis of x at the n neutral points, and the ordinates will express both in magnitude and name, the corresponding electrical intensities; in like manner let the curves $y' = P_{n-1}$, $y'' = P_{n+1}$ be constructed; if from the intersections of the two latter curves perpendiculars be let fall on the axis of x, they will determine the points of maximum intensity.

For by Prop. v., we have

$$P_n = 1 - \frac{n}{1} \cdot \frac{n+1}{1} \cdot t + \frac{n(n-1)}{1.2} \cdot \frac{(n+1)(n+2)}{1.2} \cdot t^2 - \&c.;$$

hence,

$$P_{n-1} = 1 - \frac{n-1}{1} \cdot \frac{n}{1} \cdot t + \frac{(n-1)(n-2)}{1.2} \cdot \frac{n.(n+1)}{1.2} \cdot t^2 - \&c.,$$

and

$$P_{n+1} = 1 - \frac{n+1}{1} \cdot \frac{n+2}{1} \cdot t + \frac{(n+1).n}{1.2} \cdot \frac{(n+2)(n+3)}{1.2} \cdot t^2 - \&c.;$$

therefore,

$$P_{n-1} - P_{n+1} = 2(2n+1)\left\{t - \frac{n}{1} \cdot \frac{n+1}{1} \cdot \frac{t^2}{2} + \frac{n(n-1)}{1.2} \cdot \frac{(n+1).(n+2)}{1.2} \cdot \frac{t^3}{3} - \&c.\right\}$$

$$= 2(2n+1) . \int_t P_n \text{ commencing when } t = 0,$$

but at the points of maximum intensity $\int_t P_n = 0$ by Prop. VII. Cor 1., hence at those points

$$P_{n-1} = P_{n+1} \text{ or } y' = y'';$$

they are therefore defined by the intersections of the above mentioned curves.

The n neutral points are indefinitely near each other, for whatever order of magnitude P_n belongs to, P_{n-1} and P_{n+1} may

be considered of the same order, and since $\int_t P_n = \frac{P_{n-1} - P_{n+1}}{2(2n+1)}$, it follows that $\frac{\int_t P_n}{P_n}$ is of the order $\frac{1}{n}$; but in the portion of the curve of electrical intensities between the neutral points where $x = t_m$ and $x = t_{m+1}$, this quotient is evidently of the same order as the difference $t_{m+1} - t_m$, hence the distance between two successive neutral points is of the order $\frac{1}{n}$, they are therefore indefinitely near each other, and the extreme neutral points are indefinitely near the extremities of the rod.

Suppose now the curve of electrical intensities to revolve round the axis of x, the content of the solid generated is expressed by $\pi \int_t P_n^2$ from $t = 0$ to $t = 1$, that is $\frac{\pi}{2n+1}$; this solid is composed of n small solids of revolution, of which the axes are bounded by the neutral points; each of these elementary solids is therefore of the order $\frac{1}{n^2}$, and since its axis is of the order $\frac{1}{n}$, therefore its principal section at right angles to the axis is of the same order, hence the corresponding value of P_n is of the order $\frac{1}{\sqrt{n}}$; in fact the ordinate at the middle of the rod has already been found equal to $\sqrt{\left(\frac{2}{\pi n}\right)}$.

Corollary. The most general form of the electric distribution when latent, is produced by the superposition of several systems of which the type is P_n, the number of neutral points remains indefinitely great, the quantities of the opposite electricities are still equal; but if taken in equal pairs, beginning with the first cumulus, they will not necessarily be bounded at the points of maximum intensity; lastly, the names of the

electricities at the extremities of the rod will still depend on the circumstance, whether n is an odd or even number; for if as in Prop. II., we form the equation

$$u = t + hu(1-u);$$

and if we represent by U what T becomes when t is changed into u, we get by Lagrange's Theorem,

$$\textstyle\int_u U = \int_t T + hTtt' + \frac{h^2}{1.2}\cdot\frac{d}{dt}\cdot(Tt^2t'^2) + \&c.;$$

hence the accumulation $\dfrac{d^n(Tt^nt'^n)}{dt^n}$ is the coefficient of h^n in

$$\frac{d}{dt}\cdot\textstyle\int_u U \text{ or in } U\cdot\dfrac{du}{dt}, \text{ that is, in } \dfrac{U}{\{1-2h(1-2t)+h^2\}^{\frac{1}{2}}}.$$

Now if A, B are the values of T, when t is put equal to 0 and 1 respectively, then since the corresponding values of u are also 0 and 1, we get for the accumulations at the extremities, the respective coefficients of h^n in

$$\frac{A}{\{1-2h+h^2\}^{\frac{1}{2}}} \text{ and } \frac{B}{\{1+2h+h^2\}^{\frac{1}{2}}};$$

$$\text{that is, } A \text{ and } B(-1)^n;$$

whatever, therefore, may be the proper signs of A and B, the similarity or dissimilarity of the extreme electricities will be affected by the circumstance of n being an odd or even integer.

(19). *To find the law of distribution of electricity on a spherical surface, so that the total action on external points may vary as any inverse power of the distance from the centre, higher than the second.*

Make one extremity of the diameter the origin, and let ρ be the electric accumulation in the section made by a plane perpendicular to this diameter at a distance $2at$, a being the radius of the sphere, the corresponding annulus is $4\pi a^2 . \delta t$;

and if β be the distance of any point P in the produced diameter from the origin, its distance to any point in the annulus is

$$\{a^2 + \beta^2 + 2a\beta(1-2t)\}^{\frac{1}{2}};$$

hence, the value of V (Art. 14.) is in this case expressed by

$$\int_t \frac{4\pi a^2 \rho}{\{a^2 + \beta^2 + 2a\beta(1-2t)\}^{\frac{1}{2}}},$$

$$= \frac{4\pi a^2}{\beta} \int_t \{P_0 - P_1 \frac{a}{\beta} + P_2 \frac{a^2}{\beta^2} - P_3 \frac{a^3}{\beta^3} + \&c.\} . \rho \text{ by Prop. II.,}$$

let the required accumulation on the same annulus be expanded in a form

$$\rho = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3 + \&c.;$$

$$\text{hence, } V = \frac{4\pi a^2}{\beta} \{a_0 - \frac{a_1}{3} . \frac{a}{\beta} + \frac{a_2}{5} . \frac{a^2}{\beta^2} - \frac{a_3}{7} . \frac{a^3}{\beta^3} + \&c.\},$$

whence the force on the point P tending from the centre

$$= \frac{dV}{d\beta} = -4\pi a^2 \left\{\frac{a_0}{\beta_2} - \frac{2a_1}{3} . \frac{a}{\beta^3} + \frac{3a_2}{5} . \frac{a^2}{\beta^4} - \&c.\right\};$$

hence if $\rho = a_n P_n$,

$$\text{the force on } P = -4\pi . \frac{(-1)^n . (n+1)}{2n+1} . \frac{a_n . a^{n+2}}{\beta^{n+2}}.$$

COR. 1. The value of V for internal points, is

$$4\pi a \int_t \{P_0 - P_1 . \frac{\beta}{a} + P_2 . \frac{\beta^2}{a^2} - \&c.\} . a_n P_n$$

$$= \frac{4\pi a (-1)^n . a_n}{2n+1} . \frac{\beta^n}{a^n};$$

and therefore, the force tending from the centre, or $+\frac{dV}{d\beta}$ is then

$$\frac{4\pi (-1)^n . n}{2n+1} . \frac{a_n . \beta^{n-1}}{a^{n-1}}.$$

Cor. 2. The law of electrical arrangement in this instance is such, that there are n neutral lines on the surface dividing it into $n+1$ portions, which contain alternately the opposite electricities; and since the surface of a spherical annulus is proportional to the mutual distance of the planes by which it is bounded, it is obvious from the reasoning of the preceding article, that the quantities of the opposite electricities contained between one of the neutral lines and the adjoining lines of maximum accumulation, are exactly equal; lastly, the poles will be of the same or opposite names, according as n is even or odd.

Cor. 3. If we superpose several systems of which the type is P_n, as $a_0 P_0 + a_1 P_1 + a_2 P_2 + \&c.$, then the action will be,

for internal points

$$-4\pi \left\{\frac{a_1}{3} - \frac{a_2}{5}\cdot\frac{\beta}{\alpha} + \frac{a_3}{7}\cdot\frac{\beta^2}{\alpha^2} - \&c.\right\};$$

and for external points,

$$-4\pi \left\{a_0 \cdot \frac{\alpha^2}{\beta^2} - \frac{a_1}{3}\cdot\frac{\alpha^3}{\beta^3} + \frac{a_2}{5}\cdot\frac{\alpha^4}{\beta^4} - \&c.;\right.$$

and if we take only those systems in which n is an indefinitely great number, then the actions are respectively,

$$-4\pi\left(-\frac{\beta}{\alpha}\right)^{n-1} \cdot f\left(\frac{\beta}{\alpha}\right) \text{ and } -4\pi\left(-\frac{\alpha}{\beta}\right)^{n+2} \cdot f\left(\frac{\alpha}{\beta}\right),$$

the sign f denoting a rational and entire function; and since for external points $\left(\frac{\alpha}{\beta}\right)^{n+2}$, and for internal points $\left(\frac{\beta}{\alpha}\right)^{n-1}$ are indefinitely small, the arrangement is then proper to a system of latent electricity endowed with opposite poles, or symmetrical with respect to a diameter.

(20). *To find the action of the electrical system represented by* P_n, *on any point.*

In the preceding article the system is arranged symmetrically with respect to the diameter passing through the point acted on; let now P be a point not in this line, let β be its distance from the centre, θ' the angle formed by the right line (β) which joins the centre and the point P, with the above diameter, and ϕ' the inclination of the plane in which the angle θ' lies, to a fixed plane.

In like manner let a, θ, ϕ be the polar co-ordinates of any point p in the spherical surface where the accumulation P_n is a function of θ, and a is the radius of the sphere; and putting $\cos\theta = \gamma$, we have

$$V = \int_\gamma \int_\phi \frac{a^2 P_n}{D},$$

when D equals the distance Pp;

$$\therefore\ \frac{1}{D} = \left\{a^2 - 2a\beta\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\right] + \beta^2\right\}^{-\frac{1}{2}}$$

put $\beta = ha$, and adopting the notation of Prop. VIII, we have

$$\frac{1}{D} = \frac{1}{a}\left\{Q_0 + Q_1 h + Q_2 h^2 + \&c.\right\}, \text{ for internal points,}$$

$$\text{and } \frac{1}{D} = \frac{1}{\beta}\left\{Q_0 + \frac{Q_1}{h} + \frac{Q_2}{h^2} + \&c.\right\}, \text{ for external points.}$$

Let P'_n be the value of P_n when θ' is put for θ, then by Prop. XII, we have

$$\textstyle\int_\gamma \int_\phi P_n Q_m = 0 \text{ when } m \text{ is not equal to } n,$$

$$\text{and } \textstyle\int_\gamma \int_\phi P_n Q_n = \frac{4\pi}{2n+1} \cdot P'_n.$$

Hence for internal points, we have

$$V = \frac{4\pi a}{2n+1} \cdot P'_n h^n = \frac{4\pi}{2n+1} \cdot P'_n \cdot \frac{\beta^n}{a^{n-1}},$$

and for external points,

$$V = \frac{4\pi a^2}{\beta} \cdot \frac{P'_n}{h^n} = \frac{4\pi}{2n+1} \cdot P'_n \cdot \frac{a^{n+2}}{\beta^{n+1}};$$

and the corresponding forces which are represented by $\frac{dV}{d\beta}$, are

$$\frac{4\pi \cdot n}{2n+1} \cdot P'_n \cdot \left(\frac{\beta}{a}\right)^{n-1},$$

$$\text{and } -\frac{4\pi \cdot (n+1)}{2n+1} \cdot P'_n \cdot \left(\frac{a}{\beta}\right)^{n+2};$$

that is, the electrical force follows the same law of variation as before, but the intensity is different in different diameters, and is always proportional to the electrical accumulation at the extremity of the radius which passes through the given point.

COR. 1. The values of V are both independent of ϕ', hence there is no force tending to move P out of the plane passing through that point and the fixed diameter, with respect to which the distribution is symmetrical; but in this plane, beside the force tending to or from the centre, there is another perpendicular to that direction, tending to move P circularly in that plane, its value is generally $\frac{dV}{\beta d\theta'}$, that is,

$$\text{for internal points, } \frac{4\pi}{2n+1} \cdot \frac{dP'_n}{d\theta'} \cdot \left(\frac{\beta}{a}\right)^{n-1},$$

$$\text{and for external, } \frac{4\pi}{2n+1} \cdot \frac{dP'_n}{d\theta'} \cdot \left(\frac{a}{\beta}\right)^{n+2},$$

both of which for points near the surface tend in the same direction.

COR. 2. If a series of conical surfaces which have their common vertex at the centre, be drawn through the neutral

lines, there will be no action on any points in them, tending to or from the centre; one of the surfaces is a plane passing through the centre, perpendicular to the axis of the sphere, when n is odd.

If another series of conical surfaces be drawn from the same vertex through the lines of maximum accumulation, there will be no action on points in these surfaces, which would tend to turn them round the centre; one of the surfaces is a plane when n is even.

Cor. 3. When n is indefinitely great, then $\left(\frac{a}{\beta}\right)^{n+2}$ for exterior points, and $\left(\frac{\beta}{a}\right)^{n-1}$ for interior points, are indefinitely small, hence, the electricity with the distribution represented by P_n, is completely latent; it is therefore also latent when any number of such systems are superposed.

(21). *To find the action of the latent electricity in an indefinitely thin spherical shell, on points extremely near the surface of the shell.*

First, suppose the electricity symmetrically arranged with respect to an axis, that is, to be endowed with poles; the accumulation at any point, is then represented by

$$a_n P_n + a_{n+1} P_{n+1} + a_{n+2} P_{n+2} + \&c.,$$

n being indefinitely great, using the same notation as in the last article.

Hence, for any external point, if P'_n is the particular value of P_n at the extremity of the radius drawn from the centre to that point, we have

$$V = 4\pi \frac{a^n}{\beta^n} \left\{ \frac{a_n}{2n+1} \cdot \frac{a^2}{\beta} \cdot P'_n + \frac{a_{n+1}}{2n+3} \cdot \frac{a^3}{\beta^2} \cdot P'_{n+1} + \frac{a_{n+2}}{2n+5} \cdot \frac{a^4}{\beta^3} P'_{n+2} \&c. \right\},$$

and for internal points

$$V_1 = 4\pi \frac{\beta^{n-1}}{a^{n-1}} \left\{ \frac{a_n}{2n+1} \beta . P'_n . + \frac{a_{n+1}}{2n+3} . \frac{\beta^2}{a} . P'_{n+1} \right.$$

$$\left. + \frac{a_{n+2}}{2n+5} . \frac{\beta^3}{a^2} . P'_{n+2} + \&c. \right\}.$$

Let F, ϕ be the forces which result from the first value of V, tending respectively *from*, and *round* the centre of the sphere, F_1, ϕ_1 the corresponding forces due to V_1, we have

$$F = -4\pi \frac{a^n}{\beta^n} \left\{ \frac{n+1}{2n+1} . \frac{a^2}{\beta^2} . a_n P'_n + \frac{n+2}{2n+3} . \frac{a^3}{\beta^3} . a_{n+1} P'_{n+1} \right.$$

$$\left. + \frac{n+3}{2n+5} . \frac{a^4}{\beta^4} . a_{n+2} P'_{n+2} + \&c. \right\},$$

$$F_1 = 4\pi \frac{\beta^n}{a^n} \left\{ \frac{n}{2n+1} . \frac{a}{\beta} . a_n P'_n + \frac{n+1}{2n+3} . a_{n+1} . P'_{n+1} \right.$$

$$\left. + \frac{n+2}{2n+5} . \frac{\beta}{a} . a_{n+2} P'_{n+2} + \&c. \right\},$$

$$\phi = 4\pi . \frac{a^n}{\beta^n} \left\{ \frac{1}{2n+1} . \frac{a^2}{\beta^2} . a_n . \frac{dP'_n}{d\theta'} \right.$$

$$\left. + \frac{1}{2n+3} . \frac{a^3}{\beta^3} . a_{n+1} . \frac{a P'_{n+1}}{d\theta'} + \&c. \right\},$$

$$\phi_1 = 4\pi . \frac{\beta^n}{a^n} \left\{ \frac{1}{2n+1} . \frac{a}{\beta} . a_n . \frac{dP'_n}{d\theta'} + \frac{1}{2n+3} . a_{n+1} . \frac{dP'_{n+1}}{d\theta'} + \&c. \right\}.$$

Let now Δ represent any finite distance of an external point from the surface of the shell, then $\frac{\Delta}{n}$ will be the distance of an external point which is indefinitely near the shell; β will in this case be equal to $a + \frac{\Delta}{n}$;

$$\text{and therefore, } \frac{a^n}{\beta^n} = \left(1 + \frac{\Delta}{na}\right)^{-n} = \epsilon^{-\frac{\Delta}{a}}$$

since n is indefinitely great; also $\frac{n+1}{2n+1}$ will be $= \frac{1}{2}$, $\frac{a}{\beta} = 1$, &c.

Hence for external points, at a distance $\frac{\Delta}{n}$ from the shell, we have

$$F = -2\pi\epsilon^{-\frac{\Delta}{a}}\{a_n P'_n + a_{n+1} P'_{n+1} + a_{n+2} P'_{n+2} \text{ \&c.}\}$$

$$\phi = \frac{2\pi}{n}\epsilon^{-\frac{\Delta}{a}}\left\{a_n \frac{dP'_n}{d\theta'} + a_{n+1}\frac{dP'_{n+1}}{d\theta'} + a_{n+2}\frac{dP'_{n+2}}{d\theta'} + \text{\&c.}\right\}.$$

Now the part between the first pair of brackets, expresses the electric accumulation at that point of the spherical surface which is directly *under* the point acted on; represent this accumulation by A', we then have

for external points $F = -2\pi A' . \epsilon^{-\frac{\Delta}{a}}$

$$\phi = \frac{2\pi}{n}\frac{dA'}{d\theta'} . \epsilon^{-\frac{\Delta}{a}}.$$

Similarly for internal points when $\beta = a - \frac{\Delta}{n}$,

$$F_1 = 2\pi A' \epsilon^{-\frac{\Delta}{a}}$$

$$\phi_1 = \frac{2\pi}{n} . \frac{dA'}{d\theta'} . \epsilon^{-\frac{\Delta}{a}}.$$

If now we suppose the system to be no longer symmetrical with respect to a diameter, but to result from the superposition of several systems, having different poles, it is evident that the whole accumulation at any point, will be the sum of the accumulations due to each system, and the total action will be the sum of the separate actions, when estimated *from* the centre; we shall therefore still have

$$F = -2\pi A' . \epsilon^{-\frac{\Delta}{a}}$$

$$F_1 = 2\pi A' . \epsilon^{-\frac{\Delta}{a}},$$

denoting now by A' the total accumulation on that point of the surface which is directly under the point acted on: but the forms of the expressions for the forces tending to turn the point round the centre in any plane, are to be estimated by decomposing in that plane, the force due to each separate system; and summing them, they will still be of the form

$$\phi = \frac{2\pi}{n} . A_1 . \epsilon^{-\frac{\Delta}{a}}$$

$$\phi_1 = \frac{2\pi}{n} . A_1 . \epsilon^{-\frac{\Delta}{a}},$$

A_1 being a function of the co-ordinates of the point on the surface, immediately under the point acted on.

Note. Though the expressions for ϕ, ϕ_1 contain a factor $\frac{1}{n}$ they are still of the same order with respect to magnitude, as F, F_1; in fact, if we take the elementary system expressed by $a_n P_n$, we have

$$F = -2\pi a_n P'_n . \epsilon^{-\frac{\Delta}{a}}$$

$$\phi = \frac{2\pi}{n} . a_n . \frac{dP'_n}{d\theta'} . \epsilon^{-\frac{\Delta}{a}}.$$

Now it was shewn (Art. 18.) that P'_n is of the order $\frac{1}{n^{\frac{1}{2}}}$, and $\frac{\int_t P'_n}{P'_n}$ is of the order $\frac{1}{n}$; hence, $\int_t P'_n$ is of the order $\frac{1}{n^{\frac{3}{2}}}$, and since by Prop. VII.

$$\int_t P'_n = -\frac{tt'}{n(n+1)} . \frac{dP'_n}{dt};$$

hence $\frac{dP'_n}{dt}$, and therefore also $\frac{dP'_n}{d\theta'}$ is of the order $n^{\frac{1}{2}}$;

hence, $\frac{\phi}{F}$ is of the order $\frac{\frac{1}{n} \cdot n^{\frac{1}{2}}}{\frac{1}{n^{\frac{1}{2}}}}$, that is, unity; therefore, the ratio of ϕ to F in the system resulting from superposition, will also be finite in general.

Corollary. From the preceding investigation it follows:

1. That the action of the latent electricity of a spherical shell, on points directly above the neutral points, tends solely to turn them round the centre.

2. That the actions on internal and external points, situated in the same radius, and equidistant from the shell, are equal and contrary in the direction of the radius, and equal and like in any given direction perpendicular to the radius.

3. That if a point be supposed to move parallel to the surface and extremely near it, it will be acted on alternately by attractions and repulsions, as well as by other forces tending to drive it alternately backward and forward; and the intensity of the first set of forces at a given distance, is proportional to the electrical accumulation at the point immediately under that acted on.

(22). *To find the action of the latent electricity of a plane surface on points indefinitely near it.*

Make the radius a, infinite in the expressions obtained in the last *article;* the action tending to or from the surface on any point, becomes then $2\pi\rho$, where ρ is the electrical accumulation at the point of the plane surface immediately under that acted on, it is therefore the same as the action of a plane of indefinite extent, endowed with an electrical accumulation uniformly equal to ρ, on a point situated at a finite distance from the plane.

There is also an action parallel to the surface, and which is uniform for points situated on the same normal and at

indefinitely small distances from the plane, and this is the only force which acts on points taken directly over any neutral point.

Remark. The molecules of electricity in the latent state are not to be considered in a state of absolute rest, for they are acted on by finite forces, but which rapidly pass from positive to negative; the nature of the accumulation at any instant, ought however, to be such as to satisfy the conditions that the total action on points sensibly remote, may be insensible; their motions would however, be restricted by the coercive powers exerted on them, by the material molecules of which the substance is composed.

The elementary system of electrical distribution represented by $a_n P_n$, produces by Art. 20; an action on exceedingly near points represented by $2\pi a_n P'_n \epsilon^{-\frac{\Delta}{a}}$; and that this may be a finite force, a_n must be of the order $n^{\frac{1}{2}}$, since P'_n is of the order $\frac{1}{n^{\frac{1}{2}}}$, there will also be n neutral circles on the sphere in this system, the distance between two such consecutive circles being of the order $\frac{1}{n}$; and therefore, the electricity of one kind which is accumulated on the intermediate annulus is of the order $n^{\frac{1}{2}} \times \frac{1}{n^{\frac{1}{2}}} \times \frac{1}{n}$, that is, of the order $\frac{1}{n}$, therefore the whole quantity of *either* kind of electricity in the entire indefinitely thin shell, is finite, and consequently the whole quantity of latent electricity in any solid sphere, is infinitely great; and if it be such for a sphere, it will obviously be also infinite in any body whatsoever, hence, all bodies may be conceived in their natural state, as containing an inexhaustible quantity of both electricities in a state of neutralization; and therefore, the simultaneous addition of equal but finite quantities of both electricities, will cause no difference in such phænomena as are due to the action of latent electricity.

CHAPTER IV.

ON ELECTRICITY IN A FREE STATE.

(23). *Definition of Free Electricity.*

WHEN electricity of either kind, (as for instance positive), is produced by any of the methods mentioned in Art. 1, and communicated to a perfectly conducting body, the total quantities of electricity, positive and negative, will be rendered unequal; but we may abstract from the consideration of the latent electricity, by conceiving the communicated electricity so distributed that the action on any internal point may be zero; this *excess* of either kind of electricity in bodies perfectly conducting, is called *free*, and will be equally shared by two perfectly conducting bodies of exactly the same form, when one of them is electrised and made to touch the other, so that the point of contact is similarly situated in both. Hence, free electricity differs from latent, in being continuous; that is, it does not at infinitely small distances pass from positive to negative, and as the action of the latter on all external points must be zero, in order that it may be latent, the action of the former on all internal points must also be zero, that its equilibrium may be permanent.

(24). *If* X, Y, Z *represent the total forces in the direction of the co-ordinate axes, exercised by a system of free electricity on any point* P *where the electrical density is* ρ, *and of which the co-ordinates are* x, y, z, *then shall*

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} + 4\pi\rho = 0.$$

Let a plane be drawn parallel to that of xy at a very small distance z_1 from P, and another in the same direction at a distance

a from the former, so that the point P lies in the solid bounded by both planes; if we suppose a exceedingly small, the parts of this solid in the vicinity of P, may be regarded as having the same density ρ, and the actions of this solid parallel to the bounding planes, is infinitely small compared with that which is perpendicular to them, since the force varies inversely as the square of the distance, and the density may be regarded as uniform for an extent along either of the bounding planes indefinitely greater than z_1, denote therefore by Z_1, the whole force of this solid tending to increase z; then since $2\pi\rho$ expresses the attraction of a plane of which the density is ρ, therefore the point P is attracted towards one plane with a force $2\pi\rho z_1$, and towards the other with a force $2\pi\rho(a - z_1)$; we get hence

$$Z_1 = 2\pi\rho(a - 2z_1),$$

$$\text{and } \frac{dZ_1}{dz} = -4\pi\rho \ldots\ldots\ldots (1),$$

since $z = b + z_1$, b being the distance of the lower plane from the origin.

Let now α, β, γ be the co-ordinates of any point in the solid below the lower plane, ρ' the corresponding density which is a function of those co-ordinates, and X', Y', Z' the forces resulting from this solid;

$$\text{hence, } X' = \int_\alpha \int_\beta \int_\gamma \frac{\rho'(x-\alpha)}{\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}^{\frac{3}{2}}};$$

$$\therefore \frac{dX'}{dx} = \int_\alpha \int_\beta \int_\gamma \frac{\rho'\{-2(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma^2\}}{\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}^{\frac{5}{2}}};$$

similar formulæ will be found for $\dfrac{dY'}{dy}$ and $\dfrac{dZ'}{dz}$, and none of these integrals will pass through ∞, since this solid does not contain the point P; and adding the three differential coefficients thus obtained, we have

$$\frac{dX'}{dx} + \frac{dY'}{dy} + \frac{dZ'}{dz} = 0 \ldots\ldots\ldots (2).$$

In like manner if X'', Y'', Z'' be the forces on P, due to the solid above the upper plane, and tending respectively to increase x, y, z, we have

$$\frac{dX''}{dx} + \frac{dY''}{dy} + \frac{dZ''}{dz} = 0 \ldots\ldots\ldots (3);$$

having thus considered separately the three portions into which the solid has been divided, it is evident that

$$X = X' + X'',$$
$$Y = Y' + Y'',$$
$$Z = Z' + Z'' + Z_1;$$

and adding the equations (1), (2), (3), we have

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} + 4\pi\rho = 0.$$

Corollary. When the point acted on is without the system, it follows that

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

since in this case, the density of the solid enclosed by the parallel planes, is zero.

(25). *All the free electricity with which a conducting body is charged, resides on the surface only.*

For if there were free electricity at any internal point P, if we represent its density by ρ, we have by the preceding article,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} + 4\pi\rho = 0.$$

But for all internal points, X, Y, Z must be each zero; therefore, $\rho = 0$, that is, the whole free electricity resides upon the surface, where its escape is prevented by the atmosphere, which when dry, is a non-conductor, and against which the electricity exercises a pressure.

(26). *When an electrised body is surrounded by any non-conducting medium, the pressure sustained by that medium, is proportional to the square of the electrical accumulation, at each point of the surface of the body.*

The electrical stratum which is prevented from escaping by the surrounding medium, lies on the surface of the body, and may be conceived as extending through an exceedingly small breadth into the interior; and since its interior surface is free, the resultant of all the forces which act on any point on that surface, must be directed in the normal; but with respect to the exterior surface, the tangential forces acting on a very small canal taken arbitrarily along that surface, make equilibrium with the difference of the pressures at the extremities of that canal, which being very small, the tangential force at any point is evidently a very small quantity of the second order, these forces in estimating the normal pressure, may therefore be disregarded.

Take therefore the normal at any point of the interior surface as axis of x, and the point at which it meets the exterior surface as origin, and representing by ρ the electrical density, and neglecting very small quantities of the second order, we have by Art. 24,

$$\frac{dX}{dx} + 4\pi\rho = 0;$$

and by the equation for the equilibrium of fluids, if p represent the pressure at any point at a distance x from the origin, we have

$$\frac{dp}{dx} = \rho X.$$

$$\text{Hence,}\quad p = X\textstyle\int_x\rho - \int_x\left\{\frac{dX}{dx}\int_x\rho\right\},$$

but at the interior surface $X = 0$, and at the exterior from which the integrals commence, $\int_x \rho = 0$; therefore, between the limits

$$p = -\int_x \left\{\frac{dX}{dx} \cdot \int_x \rho\right\};$$

putting now for $\frac{dX}{dx}$ its value $-4\pi\rho$, and integrating, we have

$$p = 2\pi \cdot \left\{\int_x \rho\right\}^2;$$

but $\int_x \rho$ taken between the proper limits, is the accumulation of electricity on that point of the surface, therefore the pressure is proportional to the square of the accumulation.

(27). *To find the electrical distribution on a conducting body of a spherical form.*

Let E be the whole quantity of free electricity communicated, then, since the body is perfectly symmetrical, the electricity will be spread uniformly on the surface, and will then be in a state of permanent equilibrium, since the total action on any particle of the latent electricity in the interior will then be evidently zero; hence, if a be the radius of the sphere, the surface is $4\pi a^2$, and consequently, the accumulation of electricity at each point is $\frac{E}{4\pi a^2}$, and the pressure against the surrounding medium is

$$\frac{E^2}{8\pi a^4};$$

and therefore, for equal spheres, it is proportional to the square of the total electrical charge.

(28). *To find the distribution of free electricity on the surface of a paraboloid of indefinite extent.*

Conceive an interior paraboloid exactly equal to the former, and having its axis in the same direction with the axis of the first, but its vertex at a distance a below it, if the intermediate space be filled up with homogeneous matter, the particles of which attract with forces varying inversely as the square of the distance, the total attraction on any in-

ternal point P, will be zero, for if we draw any chord through P, terminated at both sides by the exterior paraboloid, the diametral plane passing through this chord, intersects both surfaces in parabolas, which are respectively equal to the generating parabolas, and therefore also to each other; the chord which lies in the plane of these parabolas, is the double ordinate with reference to a common diameter in either, therefore the portions of the chord intercepted at opposite sides between both surfaces, are equal; if therefore, an indefinite number of such chords be drawn through P, forming two opposite conical surfaces containing a very small vertical angle, the attractions of the opposite intercepted frusta are equal and in contrary directions, and therefore mutually destroy each other; thus the effect on P of any element of the solid intercepted between the two surfaces, is destroyed by the action of an opposite element, neglecting the action of the element which is at the extremity of the diameter passing through P, and being of the same order as that element is infinitely small.

Hence, when the electrical accumulation at each point of the given paraboloid is proportional to the normal breadth of the stratum between both surfaces, there will be no action on any interior point P: now this breadth is inversely proportional to the perpendicular from the focus on the tangent plane; the pressure against the surrounding atmosphere is therefore inversely proportional to the square of this perpendicular, that is, the pressure at any point varies inversely as the distance of that point from the focus, and is therefore greatest at the vertex.

(29). *To find the distribution of free electricity on the surface of an ellipsoid.*

Conceive another ellipsoid similar and concentric with the former, and having its principal axis in the same directions with those of the given ellipsoid, to be inscribed, and let the intermediate space be filled up with homogeneous matter, the

particles of which exert forces varying inversely as the square of the distance; then it is easily proved as in the preceding article, that the total attraction on an internal point is zero, and consequently the proper distribution of electricity must be such that the accumulation at each point is as the normal breadth of this stratum, that is, as the projection of the part of the radius vector drawn from the centre, and intercepted between the surfaces, on the normal; now, by similar figures, this will be proportional to the projection of the radius vector itself, on the normal, that is, to the perpendicular from the centre on the tangent plane, and the pressure against the atmosphere is therefore as the square of this perpendicular, and consequently at the extremities of the principal axes, it is as the squares of the axes themselves.

Cor. 1. Let the ellipsoid be a prolate spheroid, and diminish indefinitely its axis minor, the axis major remaining constant, it will then be a rod of which the breadth is every where inconsiderable, but at different points varies as the mean proportional between the distances from both extremities; the tangent plane at any point not very near the extremities, is very little inclined to the axis major, the electrical accumulation, therefore, on points near the middle of the rod, is very small, and nearly uniform; but the pressure against the atmosphere at either extremity, is to that at the middle of the rod, as the square of the axis major to the square of the axis minor, it is therefore incomparably greater at the extreme points of the rod than at the middle, and consequently when the rod is surcharged with electricity, the spark will proceed from one of the extreme points, and the least circumstance will be sufficient to decide from which extremity of the rod the spark will emanate.

Cor. 2. But if the axis minor remain constant and the axis major be indefinitely increased, the form of the spheroid tends to become cylindrical, hence the accumulation and pressure may be regarded as uniform on a cylinder, of which the

length is great compared with the breadth, except at points situated near the extremities of the cylinder.

Cor. 3. If the ellipsoïd be an oblate spheroid, and while the axis major remains constant, the axis minor be indefinitely diminished, it becomes a circular disc, the thickness of which is every where inconsiderable, and at different points, varies as the mean proportional between the greatest and least distances from the circumference of the disc, and the pressure at the border of the disc is to that at its middle, as the square of the axis major to the square of the axis minor; if on the other hand the axis minor remain constant while the axis major is indefinitely increased, the spheroid becomes a solid, comprised between two parallel planes of indefinite extent, and the electrical distribution becomes uniform; in fact, the attraction of the upper plane, on points within the solid, is constant, and that of the lower is also constant, and equal to the former, and consequently the opposite and equal actions of both planes on interior points destroy each other, and will not therefore separate the latent electricities of the solid.

(30). *To find the distribution of free electricity, on any body of which the form is nearly spherical.*

Make the centre of gravity of the body, (considered homogeneous) the origin, and let a be the radius of the sphere equal in capacity to the body; let r be the radius vector of any point in the surface, θ the angle included between r and a fixed axis, and ϕ the inclination of the plane in which θ lies to a fixed plane passing through that axis; and since r differs but little from a, we may put $r = a(1 + \alpha t)$, where t is a quantity proportional to the thickness of the stratum intercepted between the surface of the sphere, and that of the body, (and may be positive or negative), and α is a very small quantity of which the squares and higher powers may be neglected.

Put $\cos\theta = \gamma$, and let t be expanded in a series

$$t = T_0 + T_1 + T_2 + T_3 + \&c.$$

the general term (T_n) of which, satisfies the equation,

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{d\,T_n}{d\gamma}\right\}+\frac{1}{1-\gamma^2}\cdot\frac{d^2\,T_n}{d\phi^2}+n\,.\,(n+1)\,.\,T_n=0,$$

the first term T_0 will in this case vanish; for the content of the whole solid considered homogeneous, is

$$\textstyle\int_r\int_\theta\int_\phi r^2\sin\theta,\ \text{ or }\ \int_r\int_\gamma\int_\phi r^2,$$

r being extended from the origin to the surface, γ from -1 to $+1$, and ϕ from 0 to 2π; the sign of the integral remains positive in the second expression, for the limits of θ are 0 and π, and that order is inverted in the limits assigned to γ, at the same time $\sin\theta=-\dfrac{d\gamma}{d\theta}$.

Integrate with respect to r, from $r=0$ to $r=a\,(1+a t)$, neglecting powers of a above the first; hence, the content is expressed by

$$\frac{a^3}{3}\textstyle\int_\gamma\int_\phi(1+3at)$$

$$=a^3\textstyle\int_\gamma\int_\phi\left(\frac{1}{3}+a\,T_0+a\,T_1+a\,T_2+\&\text{c.}\right)$$

now since $\int_\gamma\int_\phi T_n=0$ in all cases except when $n=0$, (for T_0 is constant by Prop. XI., and $\int_\gamma\int_\phi T_0T_n=0$ by Prop. XII.) hence the content is $4\pi a^3\left(\frac{1}{3}+a\,T_0\right)$; but the content of the equicapacious sphere is $\dfrac{4\pi a^3}{3}$, hence $T_0=0$.

Now if the solid were an exact sphere, the accumulation at each point would be uniform, and in the present case, as the body is nearly spherical, we may conceive the electrical action similar to that of a homogeneous stratum of nearly uniform thickness, its accumulation therefore at any point of the surface may be represented by $\rho=c\,(1+av)$, since a is very small; v is a function of θ and ϕ, which it remains to determine.

Using the notation of Art. 14, we have

$$V = \int_\gamma \int_\phi \frac{r^2 . c . (1 + av)}{\{r'^2 - 2rr'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')] + r^2\}^{\frac{1}{2}}},$$

r', θ', ϕ' being the polar co-ordinates of any point P taken within the body, and measured in the same manner that r, θ, ϕ are for any point in the surface.

Now since the whole action on an internal point must vanish, V must be constant for all such points, but using the notation of Prop. VIII, and expanding the denominator according to the ascending powers of $\frac{r'}{r}$, we have also

$$V = c \int_\gamma \int_\phi r (1 + a . v) \left\{1 + Q_1 \frac{r'}{r} + Q_2 . \frac{r'^2}{r^2} + Q_3 . \frac{r'^3}{r^3} + \&c.\right\}$$

and since this must be independent of r', we must have

$$\int_\gamma \int_\phi Q_n . r^{-(n-1)} (1 + av) = 0,$$

which condition will also render it independent of θ' and ϕ'.

Put for $r^{-(n-1)}$, its approximate value $a^{-(n-1)}\{1 - (n-1) . at\}$, and observing that $\int_\gamma \int_\phi Q_n = 0$; since n is here different from unity, we have

$$\int_\gamma \int_\phi \{v - (n-1)\, t\}\, Q_n = 0.$$

Put for t its value, and observing that when m and n are different, then $\int_\gamma \int_\phi T_m Q_n = 0$, we have

$$\int_\gamma \int_\phi \{v - (n-1) . T_n\} . Q_n = 0;$$

this equation is evidently satisfied by making

$$v = T_2 + 2 T_3 + 3 T_4 + \&c.;$$

hence, when the equation to the surface of the body is known, and the radius vector at any point is expanded in a form

$$r = a\{1 + a (T_1 + T_2 + T_3 + \ldots \&c.)\};$$

then the corresponding electrical accumulation will be

$$\rho = c\left\{1 + a\,(T_2 + 2\,T_3 + 3\,T_4 + \&c.)\right\},$$

Note. If the given body is a solid of revolution, then two polar co-ordinates r and θ will be sufficient to express the equation of the surface, θ being measured from the axis of revolution to the radius vector; and instead of taking T_n to satisfy the equation of Prop. XI, it will be then simpler to expand the thickness t in a series of quantities of the same nature as P_n (Prop. II.);

First, if r is expressed in terms of the powers of $\cos\theta$, put $\cos\theta = \mu$, it will be then necessary to expand μ^k in a series of the form $a_0 P_0 + a_1 P_1 + a_2 P_2 + \&c.$

If k is an even number as $2m$, then let

$$\mu^{2m} = a_0 P_0 + a_1 P_1 + a_2 P_2 + \ldots + a_{2m} P_{2m};$$

now P_1, P_3, &c. change their signs without altering their magnitudes when $-\mu$ is put for μ; but since μ^{2m} remains the same, we have $a_1 = 0$, $a_3 = 0$, &c., the general term is therefore of the form $a_{2n} P_{2n}$, it remains to determine a_{2n}, which by Prop. VI. $= \dfrac{(4n+1)}{2} \int_\mu P_{2n} \cdot \mu^{2m}$, the integral being taken from $\mu = -1$ to $\mu = +1$.

Now since P_{2n} contains only the even powers of μ, if we represent it by $A + B\mu^2 + C\mu^4 + \ldots\ldots N\mu^{2n}$, the integral $\int_\mu P_{2n}\mu^{2m}$ will be

$$2\left\{\frac{A}{2m+1} + \frac{B}{2m+3} + \ldots \frac{N}{2m+2n+1}\right\},$$

and this ought to vanish when m is any number of the series $0, 1, 2\ldots(n-1)$ by Prop. I, this integral is therefore of the form

$$c\,\frac{2m\,(2m-2)\,(2m-4)\ldots(2m-2n+2)}{(2m+1)\,(2m+3)\ldots(2m+2n+1)},$$

c being a constant which it remains to determine.

Multiply by $2m+1$, and then put $2m=-1$, we get thus

$$2A = c(-1)^n \cdot \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.$$

But by the third expansion of Prop. VI, we also have

$$A = \frac{1.3.5\ldots(4n-1)}{1.2.3\ldots 2n} \cdot (-1)^n \cdot \frac{2n.(2n-1)\ldots\ldots 1}{(4n-1)(4n-3)\ldots(2n+1)\times 2.4\ldots 2n}$$

$$= (-1)^n \cdot \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}, \text{ hence } c = 2;$$

hence generally,

$$\int_\mu P_{2n}\mu^{2m} = 2 \cdot \frac{2m.(2m-2)(2m-4)\ldots(2m-2n+2)}{(2m+1)(2m+3)(2m+5)\ldots(2m+2n+1)};$$

therefore,

$$a_{2n} = (4n+1) \cdot \frac{2m(2m-2)(2m-4)\ldots(2m-2n+2)}{(2m+1)(2m+3)(2m+5)\ldots(2m+2n+1)};$$

which gives

$$\mu^{2m} = \frac{P_0}{2m+1} + 5 \cdot \frac{2m.P_2}{(2m+1)(2m+3)} + 9\frac{2m(2m-2).P_4}{(2m+1)(2m+3)(2m+5)} + \&c.$$

But if k is odd as $2m+1$, then we have

$$\mu^{2m+1} = a_1 P_1 + a_3 P_3 + \ldots a_{2m+1} P_{2m+1};$$

where generally, the coefficient

$$a_{2n+1} = \frac{4n+3}{2} \cdot \int_\mu P_{2n+1} \cdot \mu^{2m+1}.$$

Now P_{2n+1} is the form $A'\mu + B'\mu^3 + C'\mu^5 + \ldots \&c.\ N'\mu^{2n+1}$, and the above integral vanishes if m be any of the numbers $0, 1, 2, 3\ldots(n-1)$, hence, as before, we get

$$\int_\mu \cdot P_{2n+1} \cdot \mu^{2m+1} = c \cdot \frac{2m(2m-2)\ldots(2m-2m+2)}{(2m+3)(2m+5)\ldots(2m+2n+3)};$$

multiply by $2m+3$ and make $2m=-3$, hence

$$2A' = c'(-1)^n . \frac{3.5\ldots(2n+1)}{2.4\ldots 2n}.$$

But by Prop. VI.

$$A' = \frac{1.3.5\ldots(4n+1)}{1.2.3\ldots(2n+1)} . (-1)^n . \frac{(2n+1)2n(2n-1)\ldots 2}{(4n+1)(4n-1)\ldots(2n+3).2.4\ldots 2n}$$

$$= (-1)^n . \frac{3.5\ldots(2n+1)}{2.4\ldots 2n}, \text{ therefore } c' = 2;$$

hence in general

$$\int\mu\, P_{2n+1} . \mu^{2m+1} = 2\, \frac{2m(2m-2)\ldots(2m-2n+2)}{(2m+3)(2m+5)\ldots(2m+2n+3)};$$

therefore,

$$\mu^{2m+1} = 3 . \frac{P_1}{2m+3} + 7 . \frac{2m . P_3}{(2m+3)(2m+5)}$$
$$+ 11 . \frac{2m . (2m-2) . P_5}{(2m+3)(2m+5)(2m+7)}.$$

Secondly, if r is expressed in terms of the cosines of the multiples of θ, put

$$\cos n\theta = a_n P_n + a_{n-2} P_{n-2} + a_{n-4} P_{n-4} + \&c.,$$

the general term of which is $a_{n-2m} P_{n-2m}$; and a_{n-2m} may be determined by a similar process applied to the fourth expansion of Prop. VI., but in this case it will be generally simpler to assume for r a series of the form $a_0 P_0 + a_1 P_1 + \ldots a_n P_n$, n being the highest number by which the arc θ is multiplied, then substituting for P_0, P_1, P_2, &c. their values given by the fourth expansion, equate the resulting formula with the given expansion of r.

CHAPTER V.

ON ELECTRICAL INFLUENCE.

(31). *To find the influence of a very remote electrised body, on a sphere also charged with electricity.*

LET a be the radius of the sphere, draw a diameter in the direction of the influencing body, as axis of x, and let that extremity of the diameter, be made origin, which is most remote from the influencing body; and let a point (p) be taken in the same diameter, at a distance β from the centre.

The action of the remote body may be considered uniform, both in magnitude and direction throughout the extent of the sphere; let this action be represented by $4\pi c$, tending to impel a particle of (suppose) positive electricity towards the origin, and let ρ be the electrical accumulation on any point of the annulus bounded by two plane sections of the sphere made at the respective distances $2at$, $2a(t+\delta t)$ from the origin, then expanding ρ in a series of the form

$$\rho = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3 + \&c.;$$

the total action of the electricity of the sphere on the point p, will be by Art. 19. Cor. 3.,

$$-4\pi\left\{\frac{a_1}{3} - \frac{2a_2}{5}\cdot\frac{\beta}{a} + \frac{3a_3}{7}\cdot\frac{\beta^2}{a^2} - \&c.\right\},$$

which tends also towards the origin; and since the particle remains in equilibrium, we have

$$c - \frac{a_1}{3} + \frac{2a_2}{5}\cdot\frac{\beta}{a} - \frac{3a_3}{7}\cdot\frac{\beta^2}{a^2} + \&c. = 0;$$

hence $a_1 = 3c$, $a_2 = 0$, $a_3 = 0$, &c.; and therefore, the accumulation is expressed by

$$\rho = a_0 P_0 + 3cP_1.$$

Let E be the total quantity of electricity with which the sphere is charged, we have

$$E = \int_t 4\pi a^2 . \rho, \text{ from } t = 0 \text{ to } t = 1;$$

put for ρ its value, and observing that $\int_t P_1 = 0$, we get

$$E = 4\pi a^2 a_0.$$

Hence the required law is expressed by

$$\rho = \frac{E}{4\pi a^2} . P_0 + 3cP_1,$$

$$\text{or, } \rho = \left(\frac{E}{4\pi a^2} + 3c\right) - 6ct.$$

Corollary. If there is a neutral line, its distance from the origin is given by the equation

$$6ct = \frac{E}{4\pi a^2} + 3c;$$

therefore the distance, $2at = \dfrac{E}{12\pi ac} + a$;

there will therefore be no neutral line if the sphere be charged either positively or negatively with a greater quantity of electricity than that which is expressed by $12\pi a^2 . c$.

When the sphere possesses only its natural electricities, $E = 0$, and then $\rho = 3cP_1 = 3c(1 - 2t)$; the neutral line is then a great circle perpendicular to the axis of x, and the two hemispheres are charged in a similar manner with the opposite electricities.

(32). *If any body not charged with electricity, but influenced by a remote electrised body, is capable of being*

divided symmetrically by a plane perpendicular to the direction of the electrical action, this plane will contain the neutral line.

Let the parts into which the body is divided by the plane, be named A and B, and let the influencing force be supposed to act horizontally, it is evident that the neutral line cannot lie in any horizontal section of the body, for there would then be no force to prevent the opposite electricities above and below this line combining; hence if the body be turned round, so that B may come into the position which A before occupied, the neutral line ought to occupy the same position in space which it did before; thus, if in its former position it lay exclusively on the portion A, it would now lie on the portion B; if in this last position we suppose the influencing force to change from positive to negative, or conversely, the neutral line would still evidently remain in the same position on B, the only difference that would thus occur, being solely that the electricities at the opposite sides of the neutral line would change their names.

Now, the change in the nature of the influencing action, has the same effect as if the force proceeded from the opposite direction without changing its nature, and the neutral line is in that case, by supposition, on A; it lies therefore, entirely on A, and also entirely on B, which is impossible unless it is the line of junction of A and B.

It is easily seen that the same reasoning would apply, if we supposed the line to lie partially on both A and B.

(33). *Any number of concentric spherical shells, the thickness of each being uniform, are separated by non-conducting media; to find the effect of their mutual influences when they are electrised.*

Let the respective quantities of electricity with which the shells are charged, be represented by $E_1, E_2, E_3 \ldots E_n$, commencing with that nearest the common centre; then since

no free electricity can reside in the space included between the interior and exterior surfaces of any given shell, (as is evident from the proof of Art. 25.) the total charge is distributed on the surfaces of the shells, and uniformly on a given shell, in consequence of the spherical forms of all the surfaces.

Let P_1, P_2, $P_3 \ldots P_n$ be a series of points taken arbitrarily in the substance of the shells, commencing as before with the inmost shell.

Now, since all the surfaces interior as well as exterior of the shells, except the interior surface of the first shell, include the point P_1 within them, their total action on that point, is zero, but the action of the electricity on the interior surface of the first shell is the same as if it were collected in the centre; now since the total force on P_1 must be zero, (Art. 23.) therefore there can be no electricity on this interior surface, consequently the charge E_1 is uniformly distributed on the exterior surface of the first shell.

Again, the effective force on P_2, results only from the exterior surface of the first shell and the interior of the second, and is the same as if all the electricity on those surfaces were collected at the centre, but since this total force must be zero, it follows that the interior surface of the second shell contains an electrical charge represented by $-E_1$, and therefore the exterior surface of the same shell must contain an electrical charge represented by $E_1 + E_2$.

In like manner it appears that the respective quantities of electricity on the inner and outer surfaces of the third shell, are $-(E_1 + E_2)$, and $E_1 + E_2 + E_3$; and in general the quantity of electricity on the inner surface of the m^{th} shell, is $-(E_1 + E_2 + E_3 \ldots E_{m-1})$, and on the outer $(E_1 + E_2 + E_3 \ldots E_m)$; that is, the quantity of electricity accumulated on the outer surface of any shell is the sum of the total charges on all the interior shells inclusive, and on the inner surface it is the same sum exclusive of the particular charge of that shell.

(34). *Two parallel plates the surface of each of which is great compared with its thickness, are separated by a thin non-conducting plate, and made to communicate, the lower with the ground, the upper with the conductor of an electrical machine; to find the effects of their mutual influences.*

Suppose, for a first approximation, that the extent of the surfaces is extremely great, and the thickness extremely small; and let a vertical right line be drawn intersecting the superior and inferior surfaces of the upper plate respectively in the points P, Q, and of the lower, in the points Q', P'; and let ρ, σ, σ', ρ' represent the corresponding electrical accumulations; then, since the extent of the plates is very great, ρ, σ, σ', may be regarded as constant, except for points situated near the edges; and $\rho' = 0$, since the lower plane is in communication with the ground.

In this vertical line which is normal to all the surfaces, take any two points p, p', the first being between the surfaces of the upper plate, the second between those of the lower; the total actions on those points must be zero.

Now, in general the action of an indefinite plane surface, of which the density is ρ, and the particles of which attract with forces which vary as the inverse square of the distance, is normal to that surface, and is represented in quantity by $2\pi\rho$.

Hence, we must have,

$$\text{for the equilibrium of } p,\ 2\pi(\sigma + \sigma' - \rho) = 0,$$

$$\text{and for } p',\ 2\pi(\sigma + \sigma' + \rho) = 0;$$

adding and subtracting these equations, we get

$$\sigma + \sigma' = 0,$$

$$\rho = 0;$$

that is, the total charge of electricity communicated to the upper plate, is distributed over its lower surface, which is in

contact with the non-conducting plate; and produces by its influence, an equal quantity of the opposite electricity, distributed in like manner over the upper surface of the plate which communicates with the ground.

This result is only approximative, since the plates are not of infinite extent; to obtain a nearer approximation, we must not consider ρ as zero, but only small compared with σ; σ' will be no longer equal to $-\sigma$, but may be represented by $-m\sigma$ where m is a quantity nearly equal to unity, and which is constant for the same system of plates, since the values of ρ, σ, σ', which are necessary for the equilibrium of p, p', may be doubled, trebled, &c., without disturbing the latent electricities at those points; let the whole charge communicated to the upper plate be represented by E.

Now since p' is in equilibrium, the action of the electrical stratum on the upper surface of the non-conducting plate, across that plate, is equivalent to the action of a stratum on the lower surface of the non-conductor, of which the accumulation would be $+m\sigma$; for we may here neglect the action of the stratum of which the accumulation is ρ, because of its greater distance from p' and the extremely small ratio of ρ to σ.

Conversely, in considering the equilibrium of p, the action of the electrical stratum of which the accumulation is $-m\sigma$, will be equivalent to that of another on the upper surface of the non-conductor, of which the accumulation would be $-m\sigma \times m$, m being a little less than unity, since the force increases when the distance diminishes in plates of which the extent is not indefinitely great.

We have now for the equilibrium of p, the equation

$$2\pi\left\{\sigma - m^2\sigma - \rho\right\} = 0;$$

$$\text{hence,}\quad \rho = +(1 - m^2)\,\sigma;$$

and if S be the surface of the plate, we have

$$S\,.\,(\rho + \sigma) = E;$$

$$\text{hence, } \sigma = \frac{1}{2 - m^2} \cdot \frac{E}{S}$$

$$\rho = \frac{1 - m^2}{2 - m^2} \cdot \frac{E}{S}$$

$$\sigma' = \frac{-m}{2 - m^2} \cdot \frac{E}{S}.$$

Corollary. If we suppose the upper plate charged until the resistance of the air just suffices to confine it to the plate,

$$\text{then } \sigma = \frac{\rho}{1 - m^2}, \text{ and } E = S\rho \cdot \frac{2 - m^2}{1 - m^2}.$$

But if when uninfluenced by the upper plate, the extreme charge which the resistance offered by the air will admit of, be represented by E', we have

$$E' = 2S\rho.$$

$$\text{Hence, } \frac{E'}{E} = \frac{2 - 2m^2}{2 - m^2},$$

and putting $m = 1 - a$ where a is very small, we get by neglecting the squares and higher powers of a,

$$\frac{E'}{E} = 4a,$$

which shews that E is much greater than E', in the ratio of $1 : 4a$, it is on this principle that condensers for accumulating great quantities of electricity are constructed; the Leyden Jar is an instance of its application.

(35). *Two opposite hyperboloids are isolated and charged with equal quantities of opposite electricities; to find the law of distribution.*

Let a pair of hyperboloids, concentric with the former and similar to them, be described, and let the intermediate space be occupied by homogeneous matter, that which lies at one

side of the centre attracting, and at the other, repelling all points with forces varying inversely as the square of the distance, the absolute forces at either side being alike; then if we take a point P, in the unoccupied space interior to either hyperbolical stratum, and form a cone, the vertical angle of which at P is extremely small, the portions of its axes intercepted by both strata, are equal, in consequence of the similarity of the bounding surfaces, and the parts which are cut out of the two strata by the cone, have therefore equal actions on P; and since one of them is attractive while the other is repulsive, the total effect on P resulting from both, is zero; and in like manner for any portion on one hyperboloid, another may be found, the force of which counteracts that of the former, thus the total action on any internal point is always zero.

If now we make both strata indefinitely thin and substitute the opposite electricities for the attractive and repulsive matter, there will be no action tending to separate the latent electricities, and this will therefore be the proper law of distribution.

It is readily seen as in the case of the ellipsoid, that the normal breadth of the stratum is as the perpendicular drawn from the centre on the tangent plane, and the pressure, as the square of this perpendicular; they are therefore greatest at the vertices of the opposite hyperboloids.

(36). *To find the influence of a non-conducting sphere, charged symmetrically with respect to its centre, on an electrised conducting sphere.*

Let O, O' be the respective centres of the conducting and non-conducting spheres, the influence of the latter is the same as if its total electrical charge (E'), were condensed in its centre, and under this influence the electrical charge (E) of the former will be arranged on its surface symmetrically with respect to the right line OO' which joins the centre.

Let a be the radius of the conducting sphere, ρ the electrical accumulation in the section made by a plane perpendicular to OO', at a distance $2at$ from the point at which $O'O$ produced, meets the sphere a second time, and let β be the distance of a point P, in OO', and within the conducting sphere; and lastly, let c represent the mutual distance of the centres.

Then since $OP = c - \beta$, the total action of the non-conducting sphere on

$$P = \frac{E'}{(c-\beta)^2}$$

$$= \frac{E'}{c^2}\left\{1 + 2\frac{\beta}{c} + 3\frac{\beta^2}{c^2} + 4\frac{\beta^3}{c^3} + \&c.\right\};$$

and this being neutralized by the action of the conducting sphere, the latter will be represented by

$$-\frac{E'}{c^2}\left\{1 + 2\frac{a}{c}.\frac{\beta}{a} + 3\frac{a^2}{c^2}.\frac{\beta^2}{a^2} + 4\frac{a^3}{c^3}.\frac{\beta^3}{a^3} + \&c.\right\}.$$

But if the required law of electrical accumulation expressed by ρ, be expanded in a form,

$$\rho = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3;$$

then by Cor. 3. Art. 19., the action of the conducting sphere on P is also expressed by

$$-4\pi\left\{\frac{a_1}{3} - \frac{2a_2}{5}.\frac{\beta}{a} + \frac{3a_3}{7}.\frac{\beta^2}{a^2} - \frac{4a_4}{9}.\frac{\beta^3}{a^3} + \&c.\right\};$$

and equating with the former value, we get

$$a_1 = \frac{3}{4\pi}.\frac{E'}{c^2},$$

$$a_2 = \frac{5}{4\pi}.\frac{a}{c}.\frac{E'}{c^2},$$

$$a_3 = \frac{7}{4\pi}\,\frac{a^2}{c^2}.\frac{E'}{c^2};$$

therefore, $\rho = a_0 P_0 + \frac{E'}{4\pi c^2}\left\{3 P_1 - 5 \cdot \frac{a}{c} \cdot P_2 + 7 \frac{a_2}{c^2} \cdot P_3 - \&c.\right\}$

$$= a_0 P_0 + \frac{E'}{4\pi a c} \cdot \left\{3 \frac{a}{c} \cdot P_1 - 5 \frac{a^2}{c^2} \cdot P_2 + 7 \frac{a^3}{c^3} P_3 - \&c.\right\}.$$

The total electrical charge on the conducting sphere $= 4\pi a^2 \int_t \rho$, this integral being taken from $t = 0$ to $t = 1$, is $4\pi a^2 a_0 = E$;

$$\text{hence, } a_0 P_0 = a_0 = \frac{E}{4\pi a^2}; \text{ and}$$

their values given by Prop. I. we get

$$\rho = \frac{1}{4\pi a}\left\{\frac{E}{a} + \frac{E'}{c}\right\} - \frac{E'}{4\pi a c}\left\{1 - 3\frac{a}{c} P_1 + 5\frac{a^2}{c^2} P_2 - 7\frac{a^3}{c^3} P_3 + \&c.\right\}$$

$$= \frac{1}{2\pi a}\left\{\frac{E}{a} + \frac{E'}{c}\right\} + \frac{E' c^{\frac{1}{2}}}{2\pi a} \frac{d}{dc}\left\{\frac{1}{c^{\frac{1}{2}}} - \frac{a}{c^{\frac{3}{2}}} \cdot P_1 + \frac{a^2}{c^{\frac{5}{2}}} P_2 - \frac{a^3}{c^{\frac{7}{2}}} P_3 + \&c.\right\}.$$

But by Prop. II.

$$\left\{1 + 2\frac{a}{c}(1 - 2t) + \frac{a^2}{c^2}\right\}^{-\frac{1}{2}} = 1 - \frac{a}{c} \cdot P_1 + \frac{a^2}{c^2} \cdot P_2 - \frac{a^3}{c^3} \cdot P_3 + \&c.;$$

multiply by $\frac{1}{c^{\frac{1}{2}}}$, and take the differential coefficients with respect to c, the left hand member of this equation will thus give

$$-\tfrac{1}{2} \cdot \frac{c^2 - a^2}{c^{\frac{1}{2}}}\left\{c^2 + 2ac(1 - 2t) + a^2\right\}^{-\frac{3}{2}};$$

whence by substitution,

$$\rho = \frac{1}{4\pi a}\left\{\frac{E}{a} + \frac{E'}{c}\right\} - \frac{E'}{4\pi a} \cdot \frac{c^2 - a^2}{\left\{c^2 + 2ac(1 - 2t) + a^2\right\}^{\frac{3}{2}}},$$

$$\text{or, } 4\pi a \cdot \rho = \frac{E}{a} + \frac{E'}{c} - \frac{E'(c^2 - a^2)}{r^3},$$

where r represents the distance from the centre of the influencing to the annulus at which the accumulation is ρ.

COR. 1. *To find the neutral line make* $\rho = 0$,

$$\text{hence,}\quad r_1 = \left\{\frac{(c^2 - a^2)\, acE'}{E'a + cE}\right\}^{\frac{1}{3}};$$

thus if the conducting sphere had no free electricity when uninfluenced, then

$$E = 0 \text{ and } r_1 = \{c \,.\, (c - a)(c + a)\}^{\frac{1}{3}},$$

r_1 being the distance of the neutral line from the centre of the influencing sphere;

$$\text{and therefore,}\quad -4\pi ac\rho \,. = E'\left(\frac{r_1^3}{r^3} - 1\right).$$

COR. 2. *To find the quantity of electricity evolved by influence.*

$$\text{Since}\quad \rho = \frac{E}{4\pi a^2} - \frac{E'}{4\pi ac}\left\{\frac{r_1}{r^3} - 1\right\},$$

where r_1 represents the distance of any point in the neutral line from the centre of the influencing sphere when $E = 0$, and the first term $\dfrac{E}{4\pi a^2}$ is due to the electrical charge of the conducting sphere, therefore if ρ' be the part of ρ dependent on the influence, the opposite electricities produced by influence on both sides of the same neutral line, are respectively represented by

$$\int_r \rho' \frac{dS}{dr}, \text{ from } r = c - a \text{ to } r = r_1,$$

$$\text{and } \int_r \rho' \frac{dS}{dr}, \text{ from } r = r_1 \text{ to } r = c + a;$$

S representing the surface of the sphere.

$$\text{But}\quad r^2 = c^2 + a^2 + 2ac(1 - 2t),$$

$$\text{hence,}\quad \frac{dr}{dt} = -4ac;$$

$$\text{also, } \frac{dS}{dt} = 4\pi a^2,$$

$$\text{hence, } \frac{dS}{dr} = -\frac{a}{c} . r;$$

the respective quantities of electricity evolved are therefore

$$\frac{E'}{4\pi c^2} \int_r \left(r - \frac{r_1^3}{r^2} \right),$$

taken between the same limits, and if we integrate, and put for r_1^3 its value $c(c^2 - a^2)$, they are respectively

$$\frac{E'}{4\pi c^2} \left\{ \frac{3}{2} (r_1^2 - c^2) - \frac{a^2}{2} \right\} \text{ and } \frac{E'}{4\pi c^2} \left\{ \frac{3}{2} (c^2 - r_1^2) + \frac{a^2}{2} \right\},$$

which quantities are exactly equal and of contrary signs, as it is easily seen they ought to be. And since r_1 is evidently less than c, the former is of an opposite kind to E', and the latter of a like kind.

COR. 3. *To find the quantity of electricity expelled when the conducting sphere is put in contact with the ground.*

Let ρ'' be the value of the electrical accumulation, at that point of the sphere which is afterwards made to communicate with the ground, the relative positions of both spheres remaining unaltered; the effect of this communication is to render the electricity at that point latent, but the same effect would be produced if we supposed a uniform stratum, of which the accumulation at every point is represented by $-\rho''$, and this stratum on account of its uniform breadth, would not disturb the latent electricities in the interior of the conducting sphere; the total quantity of electricity thus superposed, is $-4\pi a^2 \rho''$, which is equivalent to the *absence* or expulsion of a quantity represented by $4\pi a^2 \rho''$, the latter is therefore the quantity required.

Thus if we suppose the sphere to contain only its natural electricities before contact, and that the point on its surface

most remote from the influencing sphere, communicates with the ground, we have

$$\rho'' = \frac{-E'}{4\pi ac}\left\{\frac{c(c^2-a^2)}{(c+a)^3} - 1\right\}$$

$$= + \frac{E'}{4\pi ac} \cdot \frac{3ac + a^2}{(c+a)^2};$$

the quantity therefore, which is expelled, is

$$E' \cdot \left\{\frac{a^2}{(c+a)^2} \cdot \frac{3c+a}{c}\right\}.$$

(37). *To find the influence of a non-conducting spherical shell, on a conducting sphere, the electrical arrangement being supposed symmetrical with respect to the line joining the centres.*

Let O, O', and a, a' be the centres and radii of the conducting and non-conducting spheres respectively, and let A, A' be the points in which OO' produced both ways, meets the respective spheres a second time; let ρ be the electrical accumulation on a section of the conducting sphere made at a distance $2at$, from A, by a plane perpendicular to AA'; and in like manner let ρ' be the accumulation on a similar section of the other sphere, at a distance $2a't$ from A', let V, V' denote the same quantities as in Art. (14), with reference to both spheres, when points P, P' are taken in OO', at the respective distances $OP = \beta$, $OP' = \beta'$; the integrals in both spheres are taken from $t = 0$ to $t = 1$.

Let $\beta = ha$, $\beta' = h'a'$, then we have for points within the non-conducting sphere,

$$V' = 4\pi a'^2 \cdot \int_t \frac{\rho'}{\{a'^2 + \beta'^2 + 2a'\beta'(1-2t)\}^{\frac{1}{2}}}$$

$$= \frac{4\pi a'^2}{a'} \int_t \frac{\rho'}{\{1 + h'^2 + 2h'(1-2t)\}^{\frac{1}{2}}},$$

the actual value of which may be found by expanding ρ' in functions of the same nature as P_n, and putting for the denominator its value $1 - P_1 h' + P_2 h'^2 - P_3 h'^3$, &c., and then integrating; representing this integral by $\phi(h')$, we get

$$\text{when } P' \text{ is within the sphere, } V' = \frac{4\pi a'^2}{a} \phi'(h'),$$

$$\text{when } P' \text{ is without the sphere, } V' = \frac{4\pi a'^2}{\beta'} \phi'\left(\frac{1}{h'}\right).$$

Make P' to coincide with P, and to lie within the conducting sphere, and therefore, without the non-conducting; and putting c for the distance OO', we have then $\beta + \beta' = c$;

hence at the point P,

$$V' = \frac{4\pi a'^2}{c - \beta} \phi'\left(\frac{a'}{c - \beta}\right).$$

Now the force on P tending from O, by Art. (14), is

$$\frac{dV}{d\beta} - \frac{dV'}{d\beta'}, \text{ or } \frac{d(V + V')}{d\beta};$$

and since the latent electricities must not be separated when the electrical state is permanent, we get

$$V + V' = \alpha,$$

α being a constant quantity; hence for points within the conducting sphere,

$$V = \alpha - \frac{4\pi a'^2}{c - \beta} \phi'\left(\frac{a'}{c - \beta}\right),$$

$$\text{or, } \frac{4\pi a^2}{a} \int_t \frac{\rho}{\{1 + h^2 + 2h(1 - 2t)\}^{\frac{1}{2}}} = \alpha - \frac{4\pi a'^2}{c - ha} \phi'\left(\frac{a'}{c - ha}\right);$$

and representing the integral by $\phi(h)$, we have for points within the conducting sphere,

$$V = \frac{4\pi a^2}{a} \cdot \phi(h);$$

and for points without the same sphere,

$$V = \frac{4\pi a^2}{\beta} \phi\left(\frac{1}{h}\right)$$

$$= \frac{a}{\beta} \cdot \left\{ a - \frac{4\pi a'^2}{c - \frac{a}{h}} \cdot \phi'\left(\frac{a'}{c - \frac{a}{h}}\right)\right\};$$

to determine a, suppose P at an infinite distance, the value is then the same as if the total electrical charge (E) of the conducting sphere, were collected at the centre; making therefore, $h = \infty$, and multiplying by β, we get

$$E = a\left\{a - \frac{4\pi a'^2}{c} \phi'\left(\frac{a'}{c}\right)\right\};$$

whence,

$$V = \frac{E}{\beta} + 4\pi a'^2 \cdot \frac{a}{\beta}\left\{\frac{1}{c}\phi'\left(\frac{a'}{c}\right) - \frac{\beta}{c\beta - a^2}\phi'\left(\frac{a'\beta}{c\beta - a^2}\right)\right\},$$

in which expression $\frac{\beta}{a}$ is substituted for h.

Knowing thus the value of V for external points, suppose it expanded in a form

$$V = \frac{4\pi a^2}{\beta}\left\{a_0 - \tfrac{1}{3} a_1 \cdot \frac{a}{\beta} + \tfrac{1}{5} a_2 \cdot \frac{a^2}{\beta^2} - \&c.\right\};$$

then by Art. 19, we have

$$\rho = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3 + \&c.$$

Corollary.

Put $\displaystyle\int_t \frac{\rho}{\{1 + h^2 + 2h(1 - 2t)} = \phi(h);$

then for external points,

$$V = \frac{4\pi a^2}{\beta} \cdot \phi\left(\frac{1}{h}\right);$$

and comparing this with the value of V already found, we have

$$\phi\left(\frac{1}{h}\right) = \frac{E}{4\pi a^2} + \frac{a'^2}{a}\left\{\frac{1}{c}\phi'\left(\frac{a'}{c}\right) - \frac{1}{c - \frac{a}{h}}\phi'\left(\frac{a'}{c - \frac{a}{h}}\right)\right\};$$

and therefore for internal points,

$$\begin{aligned} V &= 4\pi a\phi(h) \\ &= \frac{E}{a} + 4\pi a'^2\left\{\frac{1}{c}\phi'\left(\frac{a'}{c}\right) - \frac{1}{c - ah}\phi'\left(\frac{a'}{c - ah}\right)\right\}. \end{aligned}$$

(38). *To find the electrical arrangement of two electrised conducting spheres which mutually influence each other.*

Retaining the same notation as in the last article, let the respective spheres be represented by A and A', and let $4\pi a\phi(h)$ be the value of V for points within A; the effect of the influence of A on A', is the same as if A were suddenly to become a non-conductor; therefore by the preceding Corollary, we have

$$V' = 4\pi a'\phi'(h') = \frac{E'}{a'} + 4\pi a^2\left\{\frac{1}{c}\phi\left(\frac{a}{c}\right) - \frac{1}{c - a'h'}\phi\left(\frac{a}{c - a'h'}\right)\right\}.$$

Conversely, A' may be supposed to become a non-conductor; therefore,

$$V = 4\pi a\phi(h) = \frac{E}{a} + 4\pi a'^2\left\{\frac{1}{c}\phi'\left(\frac{a'}{c}\right) - \frac{1}{c - ah}\phi'\left(\frac{a'}{c - ah}\right)\right\};$$

but by means of the former equation, the function ϕ' is given in terms of the function ϕ; thus

$$4\pi a'\phi'\left(\frac{a'}{c}\right) = \frac{E'}{a'} + 4\pi a^2\left\{\frac{1}{c}\phi\left(\frac{a}{c}\right) - \frac{c}{c^2 - a'^2}\phi\left(\frac{ac}{c^2 - a'^2}\right)\right\};$$

and

$$4\pi a'\phi'\left(\frac{a'}{c-ah}\right)=\frac{E'}{a'}+4\pi a^2\left\{\frac{1}{c}\phi\left(\frac{a}{c}\right)\right.$$

$$\left.-\frac{c-ah}{c^2-a'^2-cah}\phi\left(\frac{ac-a^2h}{c^2-a'^2-cah}\right)\right\};$$

hence,

$$\phi(h)=\frac{E}{4\pi a^2}+\frac{E'}{4\pi ac}+\frac{aa'}{c^2}\phi\left(\frac{a}{c}\right)-\frac{aa'}{c^2-a'^2}\phi\left(\frac{ac}{c^2-a'^2}\right)$$

$$-\frac{E'}{4\pi a(c-ah)}-\frac{aa'}{c(c-ah)}\phi\left(\frac{a}{c}\right)+\frac{aa'}{c^2-a'^2-cah}\phi\left(\frac{ac-a^2h}{c^2-a'^2-cah}\right).$$

Put for abridgment,

$$f=\frac{E}{4\pi a^2}+\frac{E'}{4\pi ac}+\frac{aa'}{c^2}\phi\left(\frac{a}{c}\right)-\frac{aa'}{c^2-a'^2}\phi\left(\frac{ac}{c^2-a'^2}\right)$$

$$g=\frac{E'}{4\pi a}+\frac{aa'}{c}\phi\left(\frac{a}{c}\right),$$

the equation becomes

$$\phi(h)-\frac{aa'}{c^2-a'^2-cah}\phi\left(\frac{ac-a^2h}{c^2-a'^2-cah}\right)=f-\frac{g}{c-ah};$$

from this equation the form of ϕ is to be determined, and the value of V being thus known, that of ρ will be determined by Art. 19.

Whatever may be the form or number of the electrised bodies, it is obvious from the preceding investigation, that the equation for determining the value of V in the interior of any of the bodies, will be a functional equation, the resolution of which even in the simplest cases, would offer very great analytical difficulties; the following principle will assist in resolving the question after an easier manner.

(39). *Principle of successive influences.*

Suppose two conducting bodies A and B to be electrised, but that the power by which they influence each other emanates *per saltum*, at finite and equal intervals of time; let a and b represent the electrical systems on the respective bodies when they undergo no influence, then by the first influence, a and b are changed into a' and b', the system a' being equivalent to the superposition of a system α, due solely to influence, on the original system a; and in like, b' being equivalent to the superposition of a system β on the system b, β being produced solely by influence.

The systems a' and b' may be now supposed to exercise a second influence, and since a' is equivalent to a and α, and b' to b and β, and by the first influence the action of a on b is counteracted by that of β on b; and the action of b on a by that of α on a, this second influence is equivalent merely to the influence of the systems α and β on each other; the alteration by this second influence may be regarded as equivalent to the superposition of two other systems α' and β'; the second electrical systems a'' and b'', are respectively equivalent to the systems a, α, α' and b, β, β', when each is condensed into one system by superposition.

Using the sign + to denote in this case superposition, and the sign = for the equivalent or resultant system, we have in general

$$a'''^{(n)} = a + \alpha + \alpha' + \alpha'' + \ldots\ldots + \alpha'''^{(n-1)}$$

$$b'''^{(n)} = b + \beta + \beta' + \beta'' + \ldots\ldots + \beta'''^{(n-1)}.$$

Now supposing the bodies A and B exterior to each other, the system α on the body A is produced by the influence of b, in the same manner as if B were suddenly to become a non-conductor, the system a is in equilibrium by the combined actions of α and b, and since the former is on the same body A, while the latter system is on a different body, it is evident that

a must be much less in quantity than b, and of a contrary sign, that is, of an opposite kind of electricity; for the same reason, β' which is produced by the influence of a, must be much less than a, and of a contrary sign, *a fortiori* β' is much less than b, but of the same sign; we have, therefore,

$$\beta' < \beta, \quad \beta''' < \beta', \quad \beta^{(v)} < \beta''', \text{ \&c.}$$

$$\beta < a, \quad \beta'' < \beta, \quad \beta^{iv} < \beta'', \text{ \&c.}$$

thus when n is very great, the systems $a'''^{(n-1)}$, $\beta'''^{(n-1)}$ produced by the n^{th} act of influence, are almost insensible; but the $(n+1)^{\text{th}}$ influence of the total systems, is exactly the same as the influence of the partial systems $a'''^{(n-1)}$ and $\beta'''^{(n-1)}$; the alteration produced therefore in the total systems would be insensible; hence the systems when n becomes infinite, are in permanent equilibrium, the same observations apply to any number of bodies.

This principle may be applied:

First. To obtain numerical approximations to the state of electrised bodies influencing each other, by calculating the effects of 4 or 5 successive acts of influence.

Secondly. To obtain the analytical expression for that state; for the consideration of a few successive influences will shew what the form of the quantity V is, and assuming a corresponding form with indeterminate coefficients, we may get the form for the state of B due to the influence of A, and then the state due to the influence of B or its own *reflected* influence, comparing the form thus obtained with that assumed, the indeterminate coefficients may be found.

Example. Let two equal conducting spheres, the radius of each being a unit, and the distance of their centres 100, be charged with equal quantities of electricity which may likewise be taken for units, to find the effects of their mutual influence.

Let A and A' be the two spheres, then since $a = a' = 1$, and $E = E' = 1$, and $c = 100$, we have when there is no influence

$$\phi(h) = \frac{1}{4\pi}, \quad \phi'(h) = \frac{1}{4\pi}.$$

After the first influence exercised by B on A,

$$\phi(h) = \frac{1}{4\pi} + \frac{1}{400\pi} - \frac{1}{4\pi(100-h)},$$

which also expresses $\phi'(h)$ when A first influences B.

After the second influence,

$$\phi(h) = \frac{1}{4\pi} + \frac{1}{400\pi} + \frac{1}{10000}\left\{\frac{1}{4\pi} + \frac{1}{400\pi} - \frac{1}{4\pi\left(100 - \frac{1}{100}\right)}\right\}$$

$$- \frac{1}{9999}\left\{\frac{1}{4\pi} + \frac{1}{400\pi} - \frac{1}{4\pi\left(100 - \frac{100}{9999}\right)}\right\}$$

$$- \frac{1}{4\pi(100-h)} - \frac{1}{100(100-h)}\left\{\frac{1}{4\pi} + \frac{1}{400\pi} - \frac{1}{4\pi\left(100 - \frac{1}{100}\right)}\right\}$$

$$+ \frac{1}{9999 - 100h}\left\{\frac{1}{4\pi} + \frac{1}{400\pi} - \frac{1}{4\pi\left(100 - \frac{100-h}{9999-100h}\right)}\right\},$$

where the terms which are added by the second exercise of influence are not sensible before the ninth place of decimals, and the first influence gives

$$V = 4\pi\phi(h) = 1.01 - \frac{1}{100-h}$$

$$= 1 - \frac{h}{10000}, \text{ nearly};$$

but when $\rho = a_0 P_0 + a_1 P_1 + a_2 P_2 + \ldots$

then $V = 4\pi \left\{a_0 - \frac{a_1}{3} . h + \&c.\right\}$ by Art. 19;

hence in this instance, $a_0 = \frac{1}{4\pi}$ $a_1 = \frac{3}{4\pi} . \frac{1}{10000}$;

and therefore, $\rho = \frac{1}{4\pi} + \frac{3}{40000\pi} - \frac{3t}{20000\pi}$;

thus the accumulation at that point of A which is nearest B, is less than at that point which is most remote, by a quantity which is only the $\frac{3}{5000}$ part of the mean accumulation.

A little attention to this example or to the formula for the influence of a non-conducting on a conducting sphere, (Art. 37,) will shew that in the general case of any two spheres, the successive influences introduce only constant quantities and fractions of the form $\frac{A}{a + bh}$, where A, a, b are quantities independent of h, and that the first of those fractions is of the form $\frac{g}{c - ah}$; we may therefore put generally

$$\phi(h) = f + \frac{g}{c - ah} + \frac{A_1}{a_1 + b_1 h} + \frac{A_2}{a_2 + b_2 h} + \frac{A_3}{a_3 + b_3 h} + \&c.;$$

where A_1, a_1, b_1, &c. must be determined, so that $\phi(h)$ may satisfy the equation

$$\phi(h) = f + \frac{g}{c - ah} + \frac{aa'}{c^2 - a'^2 - cah} \phi\left(\frac{a . (c - ah)}{c^2 - a'^2 - cah}\right);$$

and giving ϕ in the right-hand member of this equation, the form above assumed, we thus have

$$\phi(h) = f + \frac{g}{c - ah} = \frac{faa'}{c^2 - a'^2 - cah} + \frac{gaa'}{c(c^2 - a^2 - a'^2) - ah(c^2 - a^2)}$$
$$+ \frac{A_1 aa'}{a_1(c^2 - a'^2) + b_1 ac - ah(ca_1 + ab_1)} + \&c.;$$

and equating the corresponding terms, we get

$$A_1 = faa', \quad A_2 = gaa', \quad A_3 = f(aa')^2, \quad A_4 = g(aa')^2,$$

$$\begin{cases} a_1 = c^2 - a'^2 \\ b_1 = -ca \end{cases},$$

$$\begin{cases} a_2 = c(c^2 - a^2 - a'^2) \\ b_2 = -a(c^2 - a^2) \end{cases},$$

$$\begin{cases} a_3 = a_1(c^2 - a'^2) + b_1 \cdot ca \\ b_3 = -a_1 \cdot ca - b_1 \cdot a^2 \end{cases},$$

$$\begin{cases} a_4 = a_2(c^2 - a'^2) + b_2 \cdot ca \\ b_4 = -a_2 \cdot ca - b_2 \cdot a^2 \end{cases},$$

&c. &c.

these quantities are formed by a simple law from each other, and the general term may easily be found explicitly by a linear equation to finite differences; the value of V for internal points is therefore

$$V = \frac{4\pi a^2}{a}\left\{f + \frac{g}{c-\beta} + \frac{faa'}{a_1 + \frac{b_1}{a}\cdot\beta} + \frac{gaa'}{a_2 + \frac{b_2}{a}\cdot\beta} + \frac{f(aa')^2}{a_3 + \frac{b_3}{a}\cdot\beta} + \text{\&c.}\right\},$$

where each term except the first has the same form as when a point or non-conducting sphere symmetrically electrised with respect to its centre, influences the conducting sphere; hence by Art. 36,

$$\rho = f + \frac{g}{c}\left\{1 - \frac{c(c^2 - a^2)}{[c^2 + 2ac(1-2t) + a^2]^{\frac{3}{2}}}\right\}$$

$$+ \frac{faa'}{a_1}\left\{1 + \frac{a_1 a(a_1^2 a^2 - b_1^2 a^2)}{[a_1^2 c^2 + 2a_1^2 ac(1-2t) + a_1^2 a^2]^{\frac{3}{2}}}\right\} + \text{\&c.}$$

and a similar expression for the accumulation on A' may be obtained by changing a into a', in the quantities a_1, b_1, &c.;

and since the total quantity of electricity on A is given by the equation $E = 4\pi a^2 \int_t \rho$, from $t = 0$ to $t = 1$, the quantities f and g are determined when E and E' are given.

(40). *If two electrised bodies touch each other, and are symmetrical with respect to the common normal passing through the point of contact, there will be no free electricity at that point.*

Make the common normal the axis of x, let X, X' represent the forces along this line, at two points within the electrical stratum (on the supposition that there is electricity at the point of contact), x and x' being the distances of those two points from the origin; and since the bodies are supposed conducting, the electricity is distributed in the same manner as if the two constituted but one body; we have, therefore, by Taylor's theorem,

$$X' = X + \frac{dX}{dx}(x' - x) + \frac{d^2X}{dx^2} \cdot \frac{(x' - x)^2}{1 \cdot 2} + \text{\&c.}$$

And by Art. 24,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} + 4\pi\rho = 0.$$

But in consequence of the symmetrical figure of the bodies, there is no force except X, on points taken along the normal,

$$\text{hence, } \frac{dX}{dx} = -4\pi\rho, \quad \frac{d^2X}{dx^2} = 0, \text{ \&c.};$$

$$\text{therefore, } (X' - X) + 4\pi\rho\,(x' - x) = 0.$$

Now, at the surfaces which bound the electrical stratum within either body, $X = 0$, $X' = 0$, because the total action on any point within an electrised body, even when infinitely near the electrical stratum, is zero;

$$\text{hence, } x' - x = 0,$$

that is, the breadth of the electrical stratum is zero, at the point of contact.

Corollary. The same reasoning will apply when the tangential forces Y, Z may be neglected in comparison of the normal force X.

Remark. The motions of electrised bodies when they are non-conductors, may be calculated on the same principles as the motions of any bodies of given form, the particles of each of which exert on the others forces varying inversely as the square of the distance.

When any of the electrised bodies are conductors, the free electricity flies, partly by its own repulsion and partly by the influence of the other bodies, to its surface, where the pressure of the air is diminished at each point, by a quantity proportional to the square of the breadth of the electrical stratum, the effect is the same as if a system of forces acted normal to the surface, and tending at each point from within to without; the equations for the motion of the centre of gravity, and of the body round that point, may be formed therefore by the common principles of dynamics.

When two spheres influence each other, they produce no motion of rotation, for the forces tending to turn either sphere round its centre of gravity, in this case will evidently destroy each other.

But in other cases, the effect of the rotation as well as translation, is to make a different distribution of electricity, so that the general expression for the normal forces becomes complicated, and the exact integration of the equations of motion is much more difficult.

The action of a large electrised body on very small light substances, may be more easily computed from the circumstance that the influence of the small bodies on the larger one may be neglected.

EXAMPLE. *To determine the motion of a small pith ball, charged with electicity, influenced by a large electrised sphere.*

Representing (as in Art. 36.) the electrical charge of the ball by E, and of the sphere by E', and observing that the influence of the ball on the sphere is so inconsiderable, that it may be neglected, the latter acts as a non-conducting sphere, and therefore, by the same article, the accumulation at any point of the ball, is given by the equation

$$\rho = \frac{1}{4\pi a}\left\{\frac{E}{a} + \frac{E'}{c} - \frac{E'(c^2 - a^2)}{r^3}\right\}.$$

Let the radius drawn through the point at which the accumulation is ρ, make an angle θ, with the right line c, which joins the centres of both spheres;

$$\text{then } r^2 = c^2 - 2ac\cos\theta + a^2.$$

$$\text{Hence,} \quad r\frac{dr}{d\theta} = + ac\sin\theta,$$

$$\text{and } \sin\theta\cos\theta = \frac{c^2 + a^2 - r^2}{2a^2c^2}\,.\,r\frac{dr}{d\theta}.$$

Now the normal force on each point of the annulus on which the accumulation is ρ, is $2\pi\rho^2$, (by Art. 26,) the surface of the same annulus is $2\pi a^2\sin\theta\,.\,\delta\theta$, and resolving all the forces in the direction of c, the whole moving force is expressed by

$$4\pi^2 a^2 \textstyle\int_\theta \rho^2 \sin\theta\cos\theta$$

taking θ between the limits 0 and π.

The accelerating force on the centre of the ball, is therefore equal to

$$\frac{2\pi^2}{Mc^2}\int_r r\rho^2 \cdot (c^2 + a^2 - r^2),$$

M being the mass of the ball.

Put $$\frac{E}{a} + \frac{E'}{c} = \alpha, \quad E'(c^2 - a^2) = b,$$

and let f be this accelerating force; hence,

$$2Mc^2a^2f = \int_r \left\{ r(c^2 + a^2 - r^2)\left(\alpha^2 - \frac{2\alpha b}{r^3} + \frac{b^2}{r^6}\right) \right\},$$

taken from $r = c - a$ to $r = a + c$.

The part of this integral which is independent of b, vanishes; for when $E' = 0$, $b = 0$, and the charge is then uniformly distributed over the ball, and therefore the pressures would produce no effect; hence,

$$2Mc^2a^2f = -2\alpha b \int_r \left(\frac{c^2 + a^2}{r^2} - 1\right) + b^2 \int_r \left(\frac{c^2 + a^2}{r^5} - \frac{1}{r^3}\right)$$

$$= -2\alpha b \left\{ (c^2 + a^2)\left(\frac{1}{c - a} - \frac{1}{c + a}\right) - 2a \right\}$$

$$+ b^2 \left\{ \frac{c^2 + a^2}{4}\left(\frac{1}{(c - a)^4} - \frac{1}{(c + a)^4}\right) - \frac{1}{2}\left(\frac{1}{(c - a)^2} - \frac{1}{(c + a)^2}\right) \right\}$$

$$= -\frac{8\alpha b a^3}{c^2 - a^2} + \frac{8b^2a^3c^3}{(c^2 - a^2)^4}$$

$$= -8\alpha E' a^3 + \frac{8E'^2a^3c^3}{(c^2 - a^2)^2}.$$

Hence,

$$Mf = -\frac{4aE'a}{c^2} + \frac{4E'^2ac}{(c^2-a^2)^2}$$

$$= -\frac{4EE'}{c^2} - \frac{4E'^2a}{c^3} + \frac{4E'^2ac}{(c^2-a^2)^2}.$$

Let v represent the velocity of the ball, then

$$Mv^2 = -2M\int_c f = 8\int_c\left\{\frac{E'E}{c^2} + \frac{aE'^2}{c^3} - \frac{E'^2ca}{(c^2-a^2)^2}\right\},$$

or, $\dfrac{M}{8}\,.\,v^2 = C - \dfrac{EE'}{c} - \dfrac{aE'^2}{2c^2} + \dfrac{E'^2a}{c^2-a^2},$

C being determined by the condition that $v = 0$, when c has its initial value, and the motion is understood to occur in a horizontal plane.

CHAPTER VI.

ON ELECTRICITY DEVELOPED BY CHEMICAL ACTION.

(41). *Continued production of Electricity.*

THE chemical compositions and decompositions of substances are in general attended with a development of electricity, consequently when continued chemical action can be maintained, a continuous flow of electricity will be produced.

Thus when a metallic substance is immersed in a saline solution, the metal is found to be negatively, the solution positively electrised.

The contact of the air with substances easily susceptible of oxidation continually develops electricity of very feeble intensity, but which may be accumulated by artificial means so as to produce all the ordinary electrical phenomena.

As chemical action produces a continuous flow of electricity, conversely, a continuous flow of electricity will produce chemical actions which would not have otherwise occurred; the following experiment due to Wollaston will exhibit this.

Let a piece of zinc be partially immersed in very diluted muriatic acid; chemical action immediately takes place, the zinc dissolves, and hydrogen is disengaged from the water of the solution.

Let now a piece of silver be also partially immersed in a different part of the vessel containing the solution, the acid not acting on the silver, the latter disengages no hydrogen from the water.

Suppose now that the continued flow of electricity produced by the chemical action on the zinc, is communicated to the silver, by bringing the metals into contact outside the solution, or by means of an interposed conductor, the silver will then act on the solution, and hydrogen will be disengaged at its surface.

The decomposition of water has been also effected by the same philosopher, by means of a series of electrical sparks, when a wire isolated in its length communicates at its extremity with that fluid. (Vid. Art. 9.)

The *quantity* of electricity produced by chemical action between given substances, will in general be proportional to the extent of surface throughout which that action occurs.

Though Volta ascribed the production of a continued flow of electricity to a different cause, yet as the construction of piles for increasing its intensity was his invention, it has in this state been denominated Voltaic electricity.

(42). *Accumulation of Voltaic electricity.*

Let a plate of zinc be placed in a horizontal trough of baked wood, which is a good non-conductor, and also a plate of copper near the former and parallel to it, but not touching it, and let the intermediate space be filled with a solution of nitric acid; the zinc which is the more oxidable metal, will be acted on by the acid, and become negatively electrised; the acid becoming positively electrised, will communicate by *contact*, positive electricity to the copper.

Let another plate of zinc be placed beyond that of copper, and soldered to it, and then a second plate of copper communicating, as before, with the zinc plate, by means of an acid solution in the intermediate cell; the contact of the second zinc plate with the first plate of copper, permits the latter to communicate its positive electricity to the zinc, this positive

electricity would be communicated to the second plate of copper, if the intermediate solution acted merely as a conductor, but that liquid has also a chemical action on the zinc, which independently of the first pair of plates would be a source of positive electricity to the acid, and therefore also to the copper, with reference to which the acid may be regarded almost in the exclusive sense of a conductor; on the other hand, the same chemical action producing negative electricity for the second zinc plate, reduces the electrical state of that plate and of the copper which is soldered to it, to zero, and the acid in its capacity of a conductor, serves to double the quantity of negative electricity in the first zinc plate, the whole system being isolated by means of the trough of baked wood. It is evident that by soldering to the second plate of copper, a third zinc plate, and then putting a third copper plate, forming with the the third zinc a cell filled with acid, the quantity of positive electricity on the third plate of copper and the first of zinc will be trebled, and if we suppose the conducting power of the acid and the non-conducting powers of the trough and the atmosphere perfect, the quantities of the respective electricities developed on the final plates of zinc and copper, are proportional to the number of cells containing the acid solution: the system of plates is then denominated an electrical pile.

If sheets of paper a little moistened were interposed instead of acid, the same results would be produced but in a much lower degree; the action of the air is necessary for dry piles, for when enclosed in an air-tight vessel, they lose their power of acting when the oxygen of the enclosed air has chemically combined with the metals.

(43). *Electrical currents.*

Suppose two metallic rods are attached respectively to the plates of zinc and copper which terminate the electrical pile described in the preceding article, the rod communicating with the zinc will be perpetually in a state of positive electricity, the

other will be permanently negative, and in either case the electrical charges are equal; bring the ends of the two rods into contact, the opposite electricities combine, when they are again developed by the action of the pile, and again combined by the contact of the rods; instead of the two rods, in this case, it is manifest that one conductor communicating at once with both ends of the pile, will equally answer the purpose of permitting the electricities which are *continually* produced by the action of the pile, to recombine continually; it is this peculiar state of electricity in Voltaic conductors, which is denominated a *current*, a *direction* is also attributed to the current; for the action of the acid supplying continually the zinc end of the pile with positive electricity, and the copper end with negative, which are permitted to combine by the interposition of the conductor, the current of positive electricity in the conductor is said to be from the zinc to the copper, the terms *current* and *direction of a current* being merely conventional, and adopted simply for convenience of language in considering the phænomena of Voltaic electricity.

(44). *General effects of the Pile.*

When a shock is received from a Leyden Jar, the opposite electricities being permitted to recombine by the interposition of the conducting parts of the body, the sensation produced is instantaneous, but when the body forms a part of the Voltaic circuit, (for instance, by immersing the fingers of each hand in vessels containing solutions in which are also immersed metallic rods communicating with the ends of the pile,) the sensation is then continuous, remaining as long as the communication of the body with the ends of the pile is uninterrupted.

When Voltaic electricity is applied to produce chemical decompositions, hydrogen and the bases are found at the negative pole, (or at the extremity of the rod communicating with the negative end of the pile), while oxygen and the acids are collected round the positive.

The physiological effects of the pile are peculiarly striking, and were those which first attracted the attention of Galvani, and afterwards of Volta, to this branch of science.

The muscular contractions in the limbs of frogs, when denuded and forming part of a Voltaic circuit, (for instance, when interposed between two plates of different metals which are brought into contact,) led to the construction of the pile, by which the most violent muscular actions may be produced in the largest animals recently killed; the process of digestion can also be maintained for some time after death, by the action of Voltaic currents, and many other remarkable effects of a similar kind are produced by the same cause. (Vid. *Cumming's Electro-dynamics*, and *Art. Galvanism*, *Encycl. Met.*)

(45). *Actions of Voltaic Conductors.*

When a metallic rod or wire is made to communicate by its extremities with the ends of a Voltaic pile, the rod ceases to act in the manner of the electrised bodies considered in the former chapters, however, two such conductors of any forms, in general, act upon each other; a few of the simplest phænomena of this kind established experimentally by Ampère, form the basis from whence Voltaic actions in the most complicated cases may be computed; for the description of the apparatus and the mode of making experiments, we must refer to Professor Cumming's Electro-dynamics.

By those experiments it appears:

First. That two currents, the directions of which are mutually at right angles, exert no actions on each other.

Secondly. That an undulating conductor may be substituted for a rectilineal one which has the same extremities, provided the deviations of the former from the latter, however numerous, may be of only small extent.

Thirdly. The actions of similar conductors on points similarly situated, are equal.

Fourthly. A closed conductor exerts no action on a circular conductor, moveable round an axis passing through its centre, perpendicularly to its plane.

From the second case, it follows that we may substitute for a small portion of a conductor, three other portions which form in continuation the three edges of a parallelipiped, of which the given portion is the diagonal, a principle analogous to the resolution of forces.

The terms *closed* and *indefinite* currents are used according as the conductor forms a closed figure as an oval, or extends indefinitely as a straight line or helix.

(46). *To determine the law of force tending to or from any element of an electrical current, when points are taken at different distances but in a given direction.*

Let δs, $\delta s'$, be the elements of two electrical currents, of which the intensities are ρ, ρ' respectively; let the right line which joins their middle points be made a unit of length, and form an angle a with any fixed right line, for instance, the axis of x, and let f be the force with which these elements act on each other, the direction of which may be regarded as being in the right line forming their middle points, then the total action of the currents on each other when resolved in the direction of the same axis, is represented by $\rho\rho' \int_s \int_{s'} f \cos a$, both integrals being taken throughout the entire lengths of the currents.

Conceive now two currents similar and similarly situated to the former, but of which the linear dimensions are r times as great as in the former case; let $\delta\sigma$, $\delta\sigma'$ be the elements in the latter which correspond to the elements δs $\delta s'$ of the former,

$$\text{that is, } \delta\sigma = r\delta s$$
$$\delta\sigma' = r\delta s',$$

the mutual distance of $\delta\sigma$ $\delta\sigma'$ will be now $= r$, and if ϕ be the force with which these elements act on each other, ϕ will only

differ from f in consequence of the change of distance, for the angular position of the elements $\delta\sigma\ \delta\sigma'$ with respect to the right line joining their middles, is the same as the position of $\delta s\ \delta s'$ with respect to the corresponding line; the total action in this case taken in the direction of the axis of x, is $\rho\rho' \int_\sigma \int_{\sigma'} \phi \cos a$, or $\rho\rho' r^2 \int_s \int_{s'} \phi \cos a$, the limits of this integral being the same as before.

But by Art. 45, the mutual action of the former currents is equal to that of the latter, hence

$$\int_s \int_{s'} \phi \cos a = \frac{1}{r^2} \int_s \int_{s'} f \cos a,$$

the latter integral is a numerical quantity, hence, it is evident that ϕ must be of the form $\frac{A}{r^2}$, A depending only on the angular positions of the elements $\delta\sigma\ \delta\sigma'$, with respect to the right line joining their middles, therefore, the law of force in a specified direction, is the inverse square of the distance.

(47). *To determine the law of force tending to or from any element of a current when points are taken at a given distance, but situated in different directions with respect to the element.*

Let the right line r which joins the middles of the elements δs, $\delta s'$, be inclined to those elements at the respective angles θ, θ', the planes of which angles are mutually inclined at an angle ϕ.

By Art. 45, we may substitute for the element $\delta s'$, three other elements forming the sides of a parallelipiped, of which $\delta s'$ is the diagonal, and let one of the sides be taken in the direction of r, one perpendicular to r in the plane of θ, and one perpendicular to this plane, the current $\delta s'$ will thus be resolved into the three rectangular currents

$$\delta s' \cos\theta',\ \delta s' \sin\theta' \cos\phi,\ \delta s' \sin\theta' \sin\phi;$$

and if we substitute in like manner for δs, the sides of a parallelogram of which it is the diagonal one in the direction of r, the other perpendicular to it and in the plane of θ, the current δs will be resolved into the two currents

$$\delta s \cos\theta, \ \delta s \sin\theta.$$

In considering the mutual actions of the latter currents and the former, we may neglect those of which the directions are at right angles, (Art. 45), and observing that the direct mutual action of δs, $\delta s'$ is $\rho\rho'\frac{A}{r^2}$, where A depends only on the angular position of the elements, that is on θ, θ' and ϕ, if we represent it by $f(a)$ when the elements are parallel and inclined to the right line joining their middles at an angle a, we have

mutual action of $\delta s \cos\theta$, $\delta s' \cos\theta' = \frac{f(0)}{r^2} \cdot \rho\rho' \delta s \delta s' \cos\theta \cos\theta'$,

and the action of $\delta s \sin\theta$, and $\delta s' \sin\theta' \cos\phi$

$$= \frac{f\left(\frac{\pi}{2}\right)}{r^2} \rho\rho' \delta s \delta s' \sin\theta \sin\theta' \cos\phi,$$

because the former are in the same direction as r, and the latter perpendicular to it; let the latter quantity $f\left(\frac{\pi}{2}\right)$ be taken as a unit, then, $f(0)$ will be a certain numerical constant which may be represented by k, the total action therefore which is the sum of the above two, is

$$\frac{\rho\rho' \delta s \delta s'}{r^2} \{k \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi\}.$$

(48). *To determine the value of the numerical constant* k *in the preceding article.*

By Art. 45, it is evident that the action of any closed conductor on a circular one in its own plane, moveable round

a fixed axis passing through its centre, is always destroyed by the resistance of that axis, consequently, the total action of the first circuit on any element of the second when resolved in the direction of that element, must be zero.

Let $\delta s'$ be any element of the circular conductor, and δs any element of the given circuit, r the mutual distance of these elements, and θ', θ the respective inclinations of the same elements to the right line r, which joins their middle points; and lastly, ϕ the mutual inclination of the planes in which the angles θ', θ respectively lie.

Hence by the preceding article, the direct mutual action of δs and $\delta s'$, is

$$\frac{\rho\rho'\,\delta s\,\delta s'}{r^2}(\sin\theta\sin\theta'\cos\phi + k\cos\theta\cos\theta').$$

Suppose now that $\delta s'$ is projected on the tangent to δs, the part of this tangent between the point of contact and the foot of the perpendicular from the middle of $\delta s'$, is

$$r\cos\theta = r\frac{dr}{ds};$$

and if another perpendicular to the same right line be drawn from the extremity of $\delta s'$, this intercepted portion will be increased by the small quantity

$$\frac{d}{ds'}(r\cos\theta)\,.\,\frac{\delta s'}{2},$$

this increment is manifestly the projection of the semi-arc $\dfrac{\delta s'}{2}$ on the tangent to δs, and therefore the cosine of the inclination of the elements δs, $\delta s'$, is

$$\frac{d\left(r\dfrac{dr}{ds}\right)}{ds'}.$$

But the same cosine is also expressed by

$$\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi;$$

hence, $r\dfrac{d^2r}{ds\,ds'} + \dfrac{dr}{ds}\cdot\dfrac{dr}{ds'} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi$;

therefore, $r\dfrac{d^2r}{ds\,ds'} = \sin\theta\sin\theta'\cos\phi$;

which being substituted in the expression for the mutual action of δs, $\delta s'$, it becomes

$$\frac{\rho\rho'\delta s\,\delta s'}{r^2}\left(r\frac{d^2r}{ds\,ds'} + k\frac{dr}{ds}\frac{dr}{ds'}\right)$$

$$= \frac{\rho\rho'\delta s\,\delta s'}{r^{k+1}}\left(r^k\frac{d^2r}{ds\,ds'} + kr^{k-1}\frac{dr}{ds}\frac{dr}{ds'}\right)$$

$$= \frac{\rho\rho'\delta s\,\delta s'}{r^{k+1}}\cdot\frac{d}{ds}\left(r^k\frac{dr}{ds'}\right).$$

The resolved part of this action in the direction of the element $\delta s'$, is

$$\frac{\rho\rho'\delta s\,\delta s'}{r^{k+1}}\cdot\cos\theta'\frac{d}{ds}(r^k\cos\theta').$$

Therefore the total force tending to turn this element round the fixed axis is

$$\rho\rho'\delta s'\int_s\frac{\cos\theta'}{r^{k+1}}\cdot\frac{d}{ds}(r^k\cos\theta')$$

$$= \tfrac{1}{2}\rho\rho'\delta s'\int_s r^{-(2k+1)}\cdot\frac{d}{ds}(r^k\cos\theta')^2,$$

integrate by parts, and observing that since the given circuit is closed, the initial and final values of r and θ' are equal,

and that the total action on $\delta s'$ ought to be zero, we have

$$(2k+1)\int_r \frac{\cos^2\theta'}{r^2} = 0;$$

this equation cannot be generally true unless $2k+1=0$, because the form of the closed circuit is perfectly arbitrary, whence we obtain $k = -\frac{1}{2}$.

Corollary. The expression for the mutual action of two elements of currents, is now

$$\frac{\rho\rho'\,\delta s\,\delta s'}{r^2}\left(\sin\theta\sin\theta'\cos\phi - \tfrac{1}{2}\cos\theta\cos\theta'\right)$$

$$= \frac{\rho\rho'\,\delta s\,\delta s'}{r^2}\left(r\frac{d^2r}{ds\,ds'} - \frac{1}{2}\frac{dr}{ds}\frac{dr}{ds'}\right)$$

$$= \rho\rho'\,\delta s\,\delta s' \,.\, r^{-\frac{1}{2}}\frac{d}{ds}\left(r^{-\frac{1}{2}}\frac{dr}{ds'}\right)$$

$$= 2\rho\rho'\,\delta s\,\delta s' \,.\, r^{-\frac{1}{2}}\frac{d^2 r^{\frac{1}{2}}}{ds\,ds'}.$$

(49). *To determine the action of any given current on an element of another, in the direction of the length of that element.*

Let $\delta s'$ be the element acted on, and δs an element of the given current, their mutual action in the direction of the line joining their middle points, is

$$\rho\rho'\,\delta s\,\delta s' \,.\, r^{-\frac{1}{2}}\frac{d}{ds}(r^{-\frac{1}{2}}\cos\theta'), \text{ by Art. 48.}$$

Multiply by $\cos\theta'$ to obtain the part of this action in the direction of $\delta s'$, and integrate with respect to s, the required action becomes

$$\tfrac{1}{2}\rho\rho'\,\delta s'\,(r^{-1}\cos^2\theta' + \textit{const.});$$

let the initial values of r and θ' be R and Θ, and their final values R_1 and Θ_1; the expression for the tangential action on $\delta s'$ is therefore

$$\tfrac{1}{2}\rho\rho'\delta s'\left(\frac{\cos^2\Theta_1}{R_1}-\frac{\cos^2\Theta}{R}\right).$$

Cor. 1. An indefinite current of any form, exerts no action in the direction of the length of a given element of another current; for in this instance, R and R_1 are both infinite.

Cor. 2. A closed current will also exercise no action in this direction, for then $R=R_1$ and $\Theta=\Theta_1$; hence in both these cases, the action must be normal to the element.

Cor. 3. The actions of currents terminated at the same points, (whatever may be their forms), on any element, are equal when estimated according to the direction of that element.

Cor. 4. The total action of any fixed current on a moveable rectilineal current, in the direction of its length, is

$$\tfrac{1}{2}\rho\rho'\int_{s'}\left(\frac{\cos^2\Theta_1}{R_1}-\frac{\cos^2\Theta}{R}\right).$$

Let p, p_1 be the perpendiculars let fall from the extremities of the fixed current on the rectilineal current,

$$\text{then } R\sin\Theta=p; \qquad \text{therefore, } \frac{dR}{d\Theta}=-\frac{p\cos\Theta}{\sin^2\Theta};$$

$$\text{also, } \frac{ds'}{dR}=-\sec\Theta, \text{ hence } \frac{ds'}{d\Theta}=\frac{p}{\sin^2\Theta};$$

$$\text{similarly, } \frac{ds'}{d\Theta_1}=\frac{p_1}{\sin^2\Theta_1};$$

hence the total action is in this case

$$\tfrac{1}{2}\rho\rho'\left\{\int_{\Theta_1}\frac{\cos^2\Theta_1}{\sin\Theta_1}-\int_{\Theta}\frac{\cos^2\Theta}{\sin\Theta}\right\}.$$

Let A, A_1 be the angles which the right lines joining one extremity of the given conductor make with the rectilineal conductor, and B, B_1 the corresponding angles when the right lines are drawn from the other extremity; then

$$\int_{\Theta}\frac{\cos^2\Theta}{\sin\Theta}=\int_{\Theta}\left(\frac{1}{\sin\Theta}-\sin\Theta\right)=\mathrm{l}\frac{\tan\frac{A_1}{2}}{\tan\frac{A}{2}}+(\cos A_1-\cos A).$$

Similarly,

$$\int_{\Theta_1}\frac{\cos^2\Theta_1}{\sin\Theta_1}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathrm{l}\frac{\tan\frac{B_1}{2}}{\tan\frac{B}{2}}+(\cos B_1-\cos B),$$

the difference between those formulæ when multiplied by $\tfrac{1}{2}\rho\rho'$, is the action of the fixed conductor, whatever may be its form on the moveable rectilineal conductors in the direction of its length.

When the rectilineal conductor extends indefinitely in one direction, then A and B both become indefinitely great, and the logarithmic parts in the preceding expressions are infinite, but their difference is finite, for

$$\mathrm{l}\frac{\tan\frac{B}{2}}{\tan\frac{A}{2}}=\mathrm{l}\frac{p_1}{p},$$

since p and p_1 are the tangents of the indefinitely small angles A and B to radii which are indefinitely great, and differ but by a finite quantity.

The total action in this case is therefore,

$$\frac{1}{2}\rho\rho'\left\{1\frac{\tan\frac{A_1}{2}}{\tan\frac{B_1}{2}}\cdot\frac{p}{p_1}+(\cos A_1-\cos B_1)\right\}.$$

If the conductor extend indefinitely in both directions, then $A_1=\pi$, $B_1=\pi$, and it is easily seen that the value of

$$\frac{\tan\frac{A_1}{2}}{\tan\frac{B_1}{2}}=\frac{p}{p_1}$$

for a similar reason; the action is reduced in this case to the simple expression $\rho\rho' 1\frac{p}{p_1}$, there is therefore no action in this direction when the perpendiculars are equal.

(50). *The action of a closed current, on an element of another current which is turned in all positions round its middle point, lies in an invariable plane.*

Make the middle point of this element ($\delta s'$) the origin of three rectangular axes, and let x, y, z be the co-ordinates of any point in the circuit of which the element is represented by δs, let α', β', γ' be the inclinations of the given element $\delta s'$ to the axes of x, y, z respectively, and X, Y, Z the forces on $\delta s'$ in the directions of the same axes; then if r be the mutual distance of the elements δs and $\delta s'$, and θ' the angle which it forms with the latter, we have

$$\cos\theta'=\frac{x}{r}\cos\alpha'+\frac{y}{r}\cos\beta'+\frac{z}{r}\cos\gamma'.$$

Now the direct action of δs and $\delta s'$, is represented by

$$\rho\rho'\,\delta s\,\delta s'\,r^{-\frac{1}{2}}\frac{d}{ds}(r^{-\frac{1}{2}}\cos\theta');$$

substitute for $\cos\theta'$ its value, and resolve the force in the direction of x, multiplying it by $\frac{x}{r}$; the part of the force in that direction is thus

$$\rho\rho'\,\delta s\,\delta s'\,.\,\frac{x}{r^{\frac{3}{2}}}\frac{d}{ds}\left\{\frac{x}{r^{\frac{3}{2}}}\cos\alpha' + \frac{y}{r^{\frac{3}{2}}}\cos\beta' + \frac{z}{r^{\frac{3}{2}}}\cos\gamma'\right\};$$

hence,

$$X = \tfrac{1}{2}\rho\rho'\,\delta s\,\delta s'\left\{\cos\alpha'\int_s \frac{d}{ds}\,.\,\frac{x^2}{r^3} + \cos\beta'\int_s \frac{x}{y}\,.\,\frac{d}{ds}\,.\,\frac{y^2}{r^3}\right.$$
$$\left. + \cos\gamma'\int_s \frac{x}{z}\,.\,\frac{d}{ds}\,.\,\frac{z^2}{r^3}\right\},$$

integrating by parts, and observing that in a closed circuit the initial and final values of x, y, z, r are the same, we get

$$X = -\tfrac{1}{2}\rho\rho'\,\delta s\,\delta s'\left\{\cos\beta'\int_s \frac{y^2}{r^3}\,.\,\frac{d}{ds}\,.\,\frac{x}{y} + \cos\gamma'\int_s \frac{z^2}{r^3}\,.\,\frac{d}{ds}\,.\,\frac{x}{z}\right\};$$

or, putting for abridgement,

$$C = \int_t \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{r^3}; \qquad B = \int_t \frac{z\dfrac{dx}{dt} - x\dfrac{dz}{dt}}{r^3};$$

$$A = \int_t \frac{y\dfrac{dz}{dt} - z\dfrac{dy}{dt}}{r^3};$$

where t is any variable of which x, y and z may be regarded as functions; it is evident that A, B, C are independent of the position of the element $\delta s'$, when we suppose that element turned round its middle point, but vary with the position of that point, or with different forms of the closed circuit; we thus obtain

$$X = C\cos\beta' - B\cos\gamma';$$

similarly, $Y = A\cos\gamma' - C\cos\alpha'$,

$$Z = B\cos\alpha' - A\cos\beta';$$

hence, $AX + BY + CZ = 0.$

Now if $R^2 = X^2 + Y^2 + Z^2,$

and $D^2 = A^2 + B^2 + C^2,$

then $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ are the cosines of the angles which the resultant of all the forces on $\delta s'$ makes with the axes, and $\frac{A}{D}, \frac{B}{D}, \frac{C}{D}$ are the cosines of the angles which an invariable right line drawn through the origin makes with the same axes; hence, the resultant is always perpendicular to this invariable right line, that is, it lies in an invariable plane of which the equation is $Ax + By + Cz = 0.$

Cor. 1. Multiply the values of X, Y, Z respectively, by $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, and add; hence

$$X\cos\alpha' + Y\cos\beta' + Z\cos\gamma' = 0,$$

which shews that the resultant is normal to the element $\delta s'$, or lies in a plane of which the equation is

$$x\cos\alpha' + y\cos\beta' + z\cos\gamma' = 0,$$

which also results from the preceding article, Cor. 1: the equations of the right line in which the resultant acts are therefore

$$\begin{cases} Ax + By + Cz = 0 \\ x\cos\alpha' + y\cos\beta' + z\cos\gamma' = 0. \end{cases}$$

Cor. 2. Let ϕ be the mutual inclination of these planes, or of the invariable right line to the given element $\delta s'$, then

$$\cos\phi = \frac{A}{D} \cdot \cos\alpha' + \frac{B}{D}\cos\beta' + \frac{C}{D}\cos\gamma'.$$

Now

$$R^2 = X^2 + Y^2 + Z^2$$

$$= (\tfrac{1}{2}\rho\rho'\,\delta s')^2 . \{A^2(\cos^2\beta' + \cos^2\gamma') + B^2(\cos^2\alpha' + \cos^2\gamma')$$
$$+ C^2(\cos^2\alpha' + \cos^2\beta') - 2AB\cos\alpha'\cos\beta' - 2AC\cos\alpha'\cos\gamma'$$
$$- 2BC\cos\beta'\cos\gamma'\};$$

put for $\cos^2\beta + \cos^2\gamma'$, its value $1 - \cos^2\alpha'$;

for $\cos^2\alpha' + \cos^2\gamma'$, put $1 - \cos^2\beta'$;

and for $\cos^2\alpha' + \cos^2\beta'$, put $1 - \cos^2\gamma'$;

we thus get

$$R^2 = (\tfrac{1}{2}\rho\rho'\,\delta s')^2\{A^2 + B^2 + C^2 - (A\cos\alpha' + B\cos\beta' + C\cos\gamma')^2\}$$
$$= (\tfrac{1}{2}\rho\rho'\,\delta s')^2\{D^2 - D^2\cos^2\phi\};$$

hence, $R = \frac{1}{2}\rho\rho'\,\delta s' . D\sin\phi$;

the resultant is therefore known both in magnitude and direction, and the value of R vanishing when $\phi = 0$, shews that the invariable right line is that position into which if the element $\delta s'$ be turned, the closed circuit will exercise no action on it.

(51). *To find the action of a plane closed conductor of very small dimensions, on an element of another conductor.*

Let the origin, as before, be placed at this element, it is only necessary to calculate the values of A, B, C in this case.

Let r_1 be the projection of r on the plane of xy, forming an angle ϕ with the axis of x; then

$$C = \int_x \frac{x\dfrac{dy}{dx} - y}{r^3} = \int_\phi \frac{r_1^{\,2}}{r^3}.$$

Now r and r_1 meet the closed conductor and its projection on the plane of xy, respectively, in two points; let their second

values be represented by $r + \Delta r$ and $r_1 + \Delta r_1$, where Δr, Δr_1 are very small quantities of which all the powers higher than the first may be neglected; then observing that if ϕ be supposed to increase through that part of the conductor which is convex to the origin, it will decrease through the concave part; we have

$$C = \int_\phi \frac{r_1^2}{r^3} - \int_\phi \frac{(r_1 + \Delta r_1)^2}{(r + \Delta r)^3} = \int_\phi \left\{ \frac{3 r_1^2 \Delta r}{r^4} - \frac{2 r_1 \Delta r_1}{r^3} \right\},$$

the limiting values of ϕ being those formed by the two tangents drawn from the origin to the closed conductor.

Now $r^2 = r_1^2 + z^2$, therefore, $r \Delta r = r_1 \Delta r_1 + z \Delta z$.

Let z_1 be the distance of the point at which the plane of the conductor cuts the axis of z from the origin, then $\dfrac{\Delta z}{\Delta r_1}$ is the tangent of the inclination of r to r_1, and is therefore equal to $\dfrac{z - z_1}{r_1}$,

$$\text{hence } \Delta z = \frac{z - z_1}{r_1} \cdot \Delta r_1, \text{ and}$$

$$\Delta r = \Delta r_1 \left(\frac{r_1}{r} + \frac{z^2 - z z_1}{r r_1} \right) = \Delta r_1 \cdot \frac{r^2 - z z_1}{r r_1};$$

$$\text{therefore, } C = \int_\phi \Delta r_1 \left\{ \frac{3 r_1 (r^2 - z z_1)}{r^5} - \frac{2 r_1}{r^3} \right\}$$

$$= \int_\phi r_1 \Delta r_1 \left\{ \frac{1}{r^3} - \frac{3 z z_1}{r^5} \right\}.$$

Since the conductor is of very small dimensions, we may regard r and z as constant in this integral, and observing that the area of the projection of the given conductor is represented by $\int_\phi r_1 \Delta r_1$, if λ be the area of the closed conductor, and a, b, c the inclinations of a perpendicular p, drawn from the origin on its plane, we have

$$z_1 = p \sec c, \text{ and } \int_\phi r_1 \Delta r_1 = \lambda \cos c;$$

$$\text{hence we get } C = \lambda\left\{\frac{\cos c}{r^3} - \frac{3pz}{r^5}\right\};$$

$$\text{similarly, } B = \lambda\left\{\frac{\cos b}{r^3} - \frac{3py}{r^5}\right\},$$

$$A = \lambda\left\{\frac{\cos a}{r^3} - \frac{3px}{r^5}\right\},$$

where to x, y, z and r are to be attributed their mean values.

Cor. 1. When the element acted on is in the plane of the conductor,

$$\text{then } A = 0,\ B = 0,\ C = \frac{\lambda}{r^3},$$

the total action on the element $\delta s'$, by Art. 50, is therefore

$$\tfrac{1}{2}\rho\rho'\,\delta s' \cdot \frac{\lambda}{r^3}.$$

Cor. 2. If another very small closed conductor, of which the area is λ', be placed at the origin and in the same plane, its action on any element δs will be

$$\tfrac{1}{2}\rho\rho'\,\delta s\,\frac{\lambda'}{r^3},$$

and to obtain the action on the whole of the given current, we may resolve the current δs into the two $r\delta\phi$ and δr, the latter in the direction of the right line joining the two currents, (which are both extremely small), the former perpendicular to that line; the actions on these elements will be perpendicular to their directions, and are respectively

$$\tfrac{1}{2}\rho\rho' \cdot \frac{\lambda'\delta\phi}{r^2} \text{ and } \tfrac{1}{2}\rho\rho'\frac{\lambda'\delta r}{r^3},$$

the latter when integrated, vanishes, since the initial and final values of r are equal, the former will be

$$\tfrac{1}{2}\rho\rho'\lambda'\int_\phi \frac{1}{r^2},$$

or distinguishing as before the convex and concave parts, it is equal to

$$\tfrac{1}{2}\rho\rho'\lambda'\left\{\int_\phi \frac{1}{r^2} - \int_\phi \frac{1}{(r+\Delta r)^2}\right\}$$

$$= \rho\rho'\lambda' \int_\phi \frac{r\,\Delta r}{r^4};$$

putting for $\int_\phi r\,\Delta r$ its value λ and attributing to r in the denominator its mean value, we get for the mutual action, which is in the same direction as the right line joining the two currents, the expression $\dfrac{\rho\rho'\lambda\lambda'}{r^4}$.

(52). *A system of very small plane currents of equal areas and intensities, are ranged at equal distances along the surface of a canal of any form, the directrix of which intersects their planes at right angles; to find the total action on any element of another current.*

Let this element ($\delta s'$) be made the origin of co-ordinates, as before, let λ be the area of one of the given currents, and $\dfrac{\lambda}{m}$ the mutual distance of two which are consecutive; let x, y, z be the co-ordinates of the point at which the directrix intersects the plane of the current, δs an element of this directrix, r the mutual distance of δs, $\delta s'$; let p the perpendicular from the origin on the plane of the current (which is parallel to the tangent at δs), be inclined to the axes at the angles a, b, c; let A, B, C denote the same as in the preceding articles, except that they are here extended for the entire system of currents, the part of A due to one current (by Art. 51.) is

$$\lambda\left\{\frac{\cos a}{r^3} - \frac{3px}{r^5}\right\}.$$

But $\cos a = \frac{dx}{ds}$ and $\frac{p}{r}$, is the cosine of the inclination of r to the tangent at δs, which is also expressed by $\frac{dr}{ds}$, the number of currents corresponding to the element δs, is $\frac{m}{\lambda} . \delta s$; the part of A due to those currents is therefore,

$$m\delta s\left\{r^{-3}\frac{dx}{ds} - 3r^{-4}\frac{dr}{ds}\right\} = m\delta s\frac{d}{ds}(xr^{-3}),$$

and the total value of A is the sum of all those parts taken throughout the whole extent of the directions; hence, if the currents are indefinitely near each other, we have

$$A = m\int_s \frac{d}{ds}(xr^{-3}).$$

Let the initial and final values of x, be x_1 and x_2; and of r, r_1 and r_2

$$\text{we get, } A = m\left\{\frac{x_2}{r_2^{\,3}} - \frac{x_1}{r_1^{\,3}}\right\};$$

$$\text{similarly, } B = m\left\{\frac{y_2}{r_2^{\,3}} - \frac{y_1}{r_1^{\,3}}\right\}$$

$$\text{and } C = m\left\{\frac{z_2}{r_2^{\,3}} - \frac{z_1}{r_1^{\,3}}\right\},$$

from whence by (Art. 50), the total action on $\delta s'$ is known.

Cor. 1. If the canal be either closed or indefinitely extended in both ways, then $A = 0$, $B = 0$, $C = 0$; and therefore, there is no action on $\delta s'$.

Cor. 2. When the canal is indefinitely extended in only one direction, then

$$A = -\frac{m}{r_1^{\,3}} . x_1, \quad B = -\frac{m}{r_1^{\,3}} . y_1, \quad C = -\frac{m}{r_1^{\,3}} . z_1;$$

the resultant (by Art. 50), is perpendicular to a right line drawn from the origin, forming angles with the axes of which the cosines are $\frac{A}{D}$, $\frac{B}{D}$, $\frac{C}{D}$ respectively, where

$$D = \sqrt{A^2 + B^2 + C^2} = -\frac{m}{r_1^3} \cdot r_1,$$

which right line in this case is evidently r_1, the resultant must also be normal to $\delta s'$, since the action of each of the closed currents is so, the direction of the resultant is therefore perpendicular to the plane passing through r_1 and $\delta s'$, and if ϕ be the inclination of r_1 to $\delta s'$, the magnitude of the resultant (by Art. 50), is

$$\tfrac{1}{2}\rho\rho'\delta s'\sqrt{A^2 + B^2 + C^2}\sin\phi, \text{ or, } -\frac{m}{2}\rho\rho'\delta s' \cdot \frac{\sin\phi}{r_1^2};$$

therefore, *the action of a uniform canal of currents indefinitely extended in one way, varies inversely as the square of the distance of its extremity from the element acted on, and directly as the sine of the angle which that distance forms with the element, and is directed perpendicularly to the plane passing through the element and the extremity of the canal.*

Cor. 3. Hence the action of any uniform canal of currents may be reduced to two forces known both in magnitude and position, for this canal may be regarded as the difference of two canals, commencing respectively at the extremities of the given one and extending indefinitely in the same direction.

(53). *To find the action of any conductor on a uniform canal of currents, indefinitely extended in one direction.*

Through the extremity of the canal, as origin, draw any fixed right line which we may now regard as axis of x, let δs be an element of the conductor, and r the distance of its

middle point from the origin, this distance being inclined at an angle α to the axis of x, and at an angle ϕ to the element, the force of the canal on this element is

$$\frac{m}{2}\rho\rho'\delta s\frac{\sin\phi}{r^2},$$

(abstracting from its sign), the direction of which is perpendicular to the plane passing through r and δs; the moment of this force round the axis of x will be obtained by resolving it perpendicularly to the plane of the angle α, and multiplying by $r\sin\alpha$, which is the distance of δs from the axis of x; if therefore, ψ be the mutual inclination of the planes of the angles α and ϕ, this moment is expressed by

$$\frac{m}{2}\rho\rho'\sin\alpha\,.\,\frac{\delta s\sin\phi\cos\psi}{r}.$$

Now $\frac{\delta s}{2}\sin\phi$ is the perpendicular from the extremity of δs on r; therefore, $\frac{\delta s}{2}\sin\phi\cos\psi$ is the projection of this on the plane of α, which is equal to $r\frac{\delta\alpha}{2}$; hence the moment of the element tending to turn the canal, or to be turned by the canal round the axis of x (since action and reaction are equal), becomes $\frac{m}{2}\rho\rho'\sin\alpha\,.\,\delta\alpha$; and therefore, if α_1, α_2 be the values of α at the ends of the conductor, the whole moment is

$$\frac{m}{2}\rho\rho'(\cos\alpha_1-\cos\alpha_2).$$

In like manner the moments round the axes of y and z may be determined.

Again, to find the whole force parallel to the axis of x, we must multiply by the cosine of the inclination of the plane

of r and δs to the plane of yz, but $\frac{\delta s}{2} . r \sin\phi$ is the sectorial area comprised by the radii drawn from the origin to the extremities of δs, and when multiplied by this cosine, it is the projected area on the plane of yz, or $\frac{1}{2}(y\delta z - z\delta y)$; the total force parallel to the axis of x is therefore

$$\frac{m}{2}\rho\rho' \int_t \frac{y\frac{dz}{dt} - z\frac{dy}{dt}}{r^3},$$

t being any quantity of which y and z are functions; and using the notation of Art. 50, we have

$$\text{the force parallel to the axis of } x = \frac{m}{2}\rho\rho' A,$$

$$\text{of } y = \frac{m}{2}\rho\rho' B,$$

$$\text{of } z = \frac{m}{2}\rho\rho' C.$$

Cor. 1. If the given conductor form a closed circuit or a series of such circuits, for instance if it forms a canal of currents, the expression for the moment round the axis of x vanishes, and this axis may be any right line drawn through the extremity of the indefinite canal, hence this extremity must be the point of application of the resultant, which will lie in a direction inclined to the co-ordinate axes at angles of which the cosines are

$$\frac{A}{D}, \frac{B}{D}, \frac{C}{D} \text{ respectively, where } D = \sqrt{(A^2 + B^2 + C^2)},$$

$$\text{and the magnitude of the resultant } = \sqrt{(A^2 + B^2 + C^2)};$$

the action on a finite canal is easily estimated by regarding it as the difference of two indefinite canals.

COR. 2. When two uniform and indefinite canals of currents act on each other, the resultant passes through the extremity of each canal, and is therefore the right line joining those extremities; its magnitude is $\frac{m}{2}\rho\rho' D$, and D varies inversely as the square of this line, hence the force may be represented by $\frac{c}{R^2}$, where R is the distance between the extremities of both canals.

When the canals are finite, there will be two forces, one attractive and the other repulsive, acting at *each* extremity of either canal, the directions of which respectively pass through the two extremities of the other.

(54). *General Observations.*

In applying the preceding theory to calculate the phænomena presented by Voltaic conductors, we must observe that the *nature* of the action (with respect to repulsion or attraction) is determined by the *direction* of the current of positive electricity, it is repulsive between those parts of two currents which are in contrary directions, and attractive when the directions of the currents are the same; thus, if we conceive an element of a current to lie in a plane which bisects perpendicularly an element of another current, in one half of the latter the current approaches to, and in the other recedes from the former, the two actions are in this case equal and contrary, and therefore destroy each other; it is this perpendicularity which is to be understood in Art. 45, an instance of which for currents of finite extent, occurs when one conductor is a circular arc and the other rectilineal and passing through the centre of the arc perpendicularly to its plane. The expression

$$\frac{\rho\rho'\,\delta s\,\delta s'}{r^2}(\sin\theta\sin\theta'\cos\phi - \tfrac{1}{2}\cos\theta\cos\theta'),$$

when θ' is a right angle, vanishes, provided $\theta = 0$ or $\phi = \frac{\pi}{2}$, that is, there will be no action between two elements δs and $\delta s'$, if the former be in the perpendicular raised from the middle of the latter, or in a plane which bisects the latter perpendicularly.

From the preceding remarks it is evident that two currents passing with equal intensities through the same conductor in opposite directions will produce no action on any other conductor.

If we suppose a plane closed conductor to be divided into small portions by right lines parallel to the axis of x, we may conceive each portion to form a closed current, because each right line will be then traversed in opposite directions, and there will only remain as effective currents those which traverse the curvilineal sides of the different portions; in like manner each of these portions may be subdivided by right lines parallel to the axis of y, and thus an indefinite number of small closed currents may be substituted for the given one and the total action calculated by Arts. 51 and 52. If two such conductors act upon each other, we may subdivide each into small plane areas, the peripheries of which are traversed by currents, the mutual action is therefore the same as if each element of one surface acted on each element of the other with a force varying inversely as the fourth power of the distance, it is evident that a force of this nature cannot produce a continued motion of rotation in either circuit, round a fixed axis; the same observation applies to the mutual action of two canals of currents of any form or magnitude.

The simplest mode of observing the actions of a canal of closed currents, is by twisting a wire in the form of a helix, containing a great number of convolutions, succeeding each other at very small intervals; for then the action of each convolution is extremely nearly identical with that of a circuit absolutely closed.

(55). *Terrestrial Currents.*

Since chemical actions, by which electricity is always developed, are continually occurring both on the surface and in the interior of the earth, and since the action of heat (as observed by Seebeck) produces currents in conducting bodies, by rendering unequal the temperatures of their remote parts; accordingly Voltaic conductors of which the centres of gravity are supported, undergo terrestrial action, analogous to that produced by a system of closed currents.

The directions of the currents due to the action of heat, would like the course of the sun, be from east to west, but modified by the heat propagated in the interior of the earth, from the equator towards the poles; those attributable to chemical action would depend on the quantity and position of the substances acted on, (as on the extent of the oceans with respect to vaporisation) and would be combined in their action with the thermo-electric currents. The effective Voltaic actions indicate the direction of the terrestrial currents to be nearly from east to west, having the north pole situated on their right.

The action of a closed current on an element of a conductor is always perpendicular to that element; hence, a rectilineal conductor which is free to move in a horizontal plane, will not be moved by the influence of the earth in the direction of its length, but at right angles to that direction; but if one extremity be fixed, a continued rotation will be produced.

It is also easily seen by Arts. 50 and 53, that the action of terrestrial currents would bring a plane conductor freely suspended by its centre of gravity, into an invariable plane, and a straight canal of currents into a position perpendicular to that plane.

CHAPTER VII.

ON ELECTRICITY IN MAGNETISED SUBSTANCES.

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(56). *Magnetic Properties.*

THE term μάγνης was applied by the Greeks, to designate such substances as possessed permanently the power of attracting iron; they are most commonly iron ores, but cobalt and nickel when completely freed from ferruginous particles, possess the same property; pure iron acquires the magnetic power of attraction when in contact with magnetic substances, which it entirely loses when they are withdrawn, in the same manner that bodies electrised by influence return to their natural state when the influencing body is removed; the compounds of iron which permanently retain magnetic properties, are generally oxides, carburets, phosphurets or sulphurets of that metal: when found native, they are usually denominated load-stones.

The faculty which magnets possess of *retaining* their power to attract iron is attributed to a coercive force, this force may be modified and even destroyed by molecular displacements in the interior of the body, for instance, by increasing its temperature to a white heat.

If two bodies of different material, but having the same form and extent of surface, be electrised by the influence of the same body, the powers of attraction or repulsion which they acquire are exactly equal; but if pure iron and nickel be magnetised by the *influence* of a magnetic body, as for example, by contact with a magnetic bar, the powers which they acquire in this state are unequal, and are therefore

dependent not on the extent of surface but on the interior structure of the bodies; the whole electrical charge of any body may be carried off by covering it with an envelop of any conducting surface which is afterwards removed, (Vid. Art. 7), but nothing analogous occurs in magnets; these bodies are therefore to be regarded as systems of particles, which possess individually magnetic properties from whatever sources, but which sources are prevented from mutual communication with each other, or with external bodies by that power (depending on the molecular structure of the body) called coercive.

This will be further illustrated by attending to another remarkable property of magnets, namely, their *polarity;* if a magnetic needle be suspended by its centre of gravity, it will not like other bodies remain indifferently in any position, but will acquire a determinate direction, the inclination of which to the horizon is denominated the *dip* of the needle, and the vertical plane passing through the needle is inclined to another through the north and south points of the horizon at a determinate angle called the variation, these angles are different at different places, and even at any given place they undergo slow changes in the progress of time: the dip at London is now about $70\frac{1}{2}^{0}$, and the variation about $24\frac{1}{2}$ degrees; that part of the needle which is turned towards the south or *from the north,* is called its north pole, and the other part the south pole of the needle: if we try to invert the position of the needle by turning the poles round, it will right itself by turning through 180^{0} of azimuth, into its original place; if the needle be rolled in fine filings of iron, they will attach themselves to it in great quantities at certain points situated near the extremities which are more particularly called the poles of the needle, and in very small quantity at the parts situated near the middle of the needle; if two magnetic needles be placed with their like poles near each other, they repel, but with the unlike they attract; if the poles be divided by breaking the needle, each of the halves will be found similarly endowed with poles, unlike poles being found at the parts where the fracture occurred; however small the fragments

into which the needle is divided, the separate parts enjoy all the same properties with respect to polarity, as the whole needle; thus the magnetic phænomena are compound, resulting from indefinitely small portions of the magnetized body, which are restrained by the coercive force from mutual communication in the same manner that electricity is retained at the surface of bodies, by the resistance which the air offers to its escape.

All iron would become permanently magnetic if it possessed this coercive power; place a bar of soft iron in the same position into which the action of the earth would draw magnetic bodies, it will become a magnet by influence, having, as the needle, north and south poles; but if we turn it round, the poles do not turn with it as in the magnetic needle, nor will it tend to turn round to its original position, but always acquires a new magnetic state corresponding to the new position into which it is moved; it only wants coercive power, to retain the magnetic state which it once acquires, to be a complete magnet.

(57). *Electro-magnetic Phænomena.*

The agitation of the compass needle during the appearance of the Aurora Borealis, and the inversion of its poles when struck with lightning, were the first natural phænomena which led to the belief that magnetism and electricity were connected in their nature.

The electrical discharge of a strong battery was found capable of rendering ordinary steel needles magnetic, and of inverting the poles of those already magnetised.

It was discovered by Oersted, that Voltaic conductors act on magnets, and conversely that magnets act on Voltaic conductors: if an electrical discharge be passed through a conductor in the form of a helix, a steel needle placed parallel to the axis of the helix and within it becomes strongly mag-

netised; lastly, a Voltaic conductor attracts iron filings to its surface while transmitting electricity, but when the transmission is discontinued, the filings immediately drop off.

The magnetism of the earth induces no motion of translation in magnetised bodies, but merely *turns* them into a particular position; it acts in the same manner on a Voltaic helix.

The action of a Voltaic conductor on a magnetic needle, presents in general the same effects as if a uniform canal of currents were substituted for the needle, and conversely: also the mutual actions of magnetic needles are analogous to those already demonstrated, to belong to canals of Voltaic currents.

A magnetic substance of which the form is annular, ceases apparently to possess magnetic action; a closed canal of currents in like manner exhibits no Voltaic action. (Art. 52. Cor. 1.)

If one magnet neither traverses nor is affixed to another, it cannot produce continued rotation in it; the same is true of any system of closed currents. (Art. 54.)

When the preceding conditions are not satisfied, continued rotation may be produced either in conductors by the action of magnets, or conversely; the action of the terrestrial currents produces continued rotation in magnetised bars or Voltaic conductors properly disposed, and lastly, the electrical spark can be drawn from magnets or magnetic ores; science is indebted to Mr Faraday for most of the latter facts.

From the analogy between the actions of magnets and Voltaic helices, M. Ampere to whom the theory of the action of electrical currents is principally due, has adopted particular views with respect to the nature of magnetised bodies, assigning as the cause of their actions the existence of closed currents circulating round their molecules.

However this may be, the preceding facts seem completely to establish an identity between electricity and magnetism, with this peculiarity, that in magnetised substances each magnetic particle is to be regarded in itself as a magnet, (Vid. Art. 56). Since a needle can be permanently magnetised merely by influence, (as by the electrical discharge through a helix), and when unmagnetised it exerts no action, it follows that the quantities of magnetism which respectively attract or repel a given magnetic molecule must be equal, (Art. 15.) they are denominated for the purpose of distinction, south and north magnetisms, the latter being that which in an element of a magnetic body, is repelled *from the north* by terrestrial magnetic action.

(58). *To find the action of a system of magnetic particles composing a magnetised body of any form, on a point containing north magnetism, and situated at any sensible distance from that body.*

Let x, y, z be the co-ordinates of the point (P) acted on, and x', y', z' of a point (P') taken within one of the magnetic elements (M), and let e represent the side of a very small cube of the same magnitude as M, then $x'+e\alpha$, $y'+e\beta$, $z'+e\gamma$ may be taken to represent the co-ordinates of any other point (p') in the same element (M), and α, β, γ will be finite.

Suppose that the excess of north magnetism at the point (p') above south is represented by ρ, the respective distances of the points P' and p' from the given point P by R and r, that as in electricity generally, the law of force is the inverse square of the distance, magnetisms of the same name repelling, and of different names attracting, then the action of p' on P in the direction of the axis of x will be expressed by $-\dfrac{d\left(\frac{1}{r}\right)}{dx}$ multiplied by the quantity of free electricity at p', that is, by $\rho e^3\,\delta\alpha\,\delta\beta\,\delta\gamma$; consequently the action of the complete element M on P in that direction will be

$$-e^3 \int_\alpha \int_\beta \int_\gamma \rho \frac{d\left(\frac{1}{r}\right)}{dx},$$

the integral being extended throughout the whole of that element.

$$\text{Now } \frac{1}{r} = \frac{1}{R} + e\alpha \frac{d\left(\frac{1}{R}\right)}{dx'} + e\beta \frac{d\left(\frac{1}{R}\right)}{dy'} + e\gamma \frac{d\left(\frac{1}{R}\right)}{dz'},$$

neglecting the powers of e which are higher than the first;

$$\text{also, } \int_\alpha \int_\beta \int_\gamma \rho \cdot \frac{d\left(\frac{1}{R}\right)}{dx} = 0,$$

since the quantities of north and south magnetisms in the element are equal; hence, the force of the element M on the point P becomes

$$-M \cdot \int_\alpha \int_\beta \int_\gamma \rho \frac{d}{dx} \left\{ e\alpha \frac{d \cdot \frac{1}{R}}{dx'} + e\beta \frac{d \cdot \frac{1}{R}}{dy'} + e\gamma \frac{d \cdot \frac{1}{R}}{dz'} \right\}$$

$$= -\frac{d}{dx} \left\{ M \frac{d\frac{1}{R}}{dx'} \int_\alpha \int_\beta \int_\gamma \rho e\alpha + M \frac{d\frac{1}{R}}{dy'} \int_\alpha \int_\beta \int_\gamma \rho e\beta \right.$$

$$\left. + M \frac{d\frac{1}{R}}{dz'} \int_\alpha \int_\beta \int_\gamma \rho e\gamma \right\}.$$

To sum this expression throughout the whole extent, conceive a parallelipiped of which the sides $\delta x'$, $\delta y'$, $\delta z'$ parallel to the axes are very small compared with the dimensions of the body, but which itself contains a very great number of magnetic elements such as M, the sum of all which elements within it may be represented by $k' \delta x' \delta y' \delta z'$, that is, in the proportion of $k' : 1$ to its entire bulk; the preceding ex-

pression will still apply for the whole parallelipiped, only putting $k'\delta x'\delta y'\delta z'$ instead of M, and substituting for the triple integrals their mean values throughout its extent, which may be regarded as functions of x', y', z', and be represented respectively by a', b', c'; the action of this parallelipiped on P estimated in the direction of x, is therefore

$$-\frac{d}{dx}\cdot k'\delta x'\delta y'\delta z'\left\{a'\frac{d\cdot\frac{1}{R}}{dx'}+b'\frac{d\cdot\frac{1}{R}}{dy'}+c'\frac{d\cdot\frac{1}{R}}{dz'}\right\};$$

hence if X, Y, Z be the total magnetic forces exercised by the whole magnetised body on P, and making

$$Q=\int_{x'}\int_{y'}\int_{z'}\left\{k'a'\frac{d\cdot\frac{1}{R}}{dx'}+k'b'\frac{d\cdot\frac{1}{R}}{dy'}+k'c'\frac{d\cdot\frac{1}{R}}{dz'}\right\},$$

this integral being taken throughout the entire body, we have

$$X=-\frac{dQ}{dx},\quad Y=-\frac{dQ}{dy},\quad Z=-\frac{dQ}{dz}.$$

(59). *To determine the action of a system of particles magnetised solely by influence, and incapable of retaining their magnetic state when the influencing force is withdrawn.*

The quantities a', b', c' evidently depend on the action of the influencing force which we first suppose constant in magnitude and position, and vanish at the same time with it; also in passing from one system of co-ordinates to another, it is manifest (from the forms of the integrals of which they are the mean values,) that they undergo changes of form exactly similar to those of the components of a given force; hence if we represent in this case the components at any given instant by f', g', h' parallel respectively to x, y and z, it follows that a' must be a function of these forces of the form

$$a'=Cf'+C'g'+C''h'.$$

Suppose now that the point P is taken in the right line drawn from P' parallel to the axis of y, and that the force acts in the same direction, that is, $f'=0$, $h'=0$, $a'=C'g'$; now the action of the small magnetic system at P' on P by the preceding article, is

$$-k'\,\delta x'\,\delta y'\,\delta z'\,\frac{d}{dx}\left\{a'\frac{d.\frac{1}{R}}{dx'}+b'\frac{d.\frac{1}{R}}{dy'}+c'\frac{d.\frac{1}{R}}{dz'}\right\},$$

putting in the present case $x'=x$ and $z'=z$ after the differentiations are performed, this becomes

$$2\,k'\,\delta x'\,\delta y'\,\delta z'\,.\,\frac{a'}{R^3},$$

which ought to be zero, since the whole action is in the direction of y, hence $a'=0$ when f' and h' vanish, which requires $C'=0$; similarly, it may be shewn that $C''=0$, therefore we get $a'=Cf'$, and consequently $b'=Cg'$, $c'=Ch'$, since the quantities a', b', c' are susceptible of the same changes relative to the axes as the forces f', g', h'.

But when the point considered (P) is in the influenced body, suppose a, b, c to be what a', b', c' become when x, y, z are put for x', y', z'; conceive a very small sphere, having P for centre, and containing a very great number of magnetic elements, throughout which we may put for k', a', b', c' their mean values as k, a, b, c, then the forces (f, g, h) result from all the extraneous forces which influence the whole system, and also from that part of the system itself, which is beyond this sphere: the part of the former forces in the direction of x, will be represented by $-\frac{dV}{dx}$, using V in the same sense as in Art. 14, and by the preceding article, the corresponding force arising from the latter source is $-\frac{dQ_1}{dx}$, Q_1 representing the

value of Q, taken for the whole magnetic system except the small sphere,

$$\text{hence, } f = -\frac{dV}{dx} - \frac{dQ_1}{dx} \text{ and } a = -C\left(\frac{dV}{dx} + \frac{dQ_1}{dx}\right).$$

Now $\frac{dQ_1}{dx}$ differs from $\frac{dQ}{dx}$ by a quantity due to the small sphere, namely, by

$$\int_{x'}\int_{y'}\int_{z'}\left\{k'a'\frac{d.\frac{x'-x}{R^3}}{dx'} + k'b'\frac{d.\frac{x'-x}{R^3}}{dy'} + k'c'\frac{d.\frac{x'-x}{R^3}}{dz'}\right\},$$

where the differentiations with respect to x have been performed under the signs of integration, since the integrals commence from $x' = x$, $y' = y$, $z' = z$, and extend throughout the entire sphere; then performing the integration with respect to x' in the first term, y' in the second, and z' in the third, and taking any two variables u and v, with respect to which the element of the surface of the sphere is $\frac{d^2S}{du\,dv}.\delta u\,\delta v$; lastly, putting θ, θ', θ'' for the inclinations of the radius passing through that element to the axes, observing that

$$\delta z'\,\delta y' = \frac{d^2S}{du\,dv}.\delta u\,\delta v\cos\theta,$$

this expression becomes

$$k\int_u\int_v\{a\cos\theta + b\cos\theta' + c\cos\theta''\}.\frac{x'-x}{R^3}.\frac{d^2S}{du\,dv},$$

where to x' and R we are to assign their values at the surface of the sphere.

Let for instance, u be the angle (θ) which R makes with the axis of x, and v the inclination of the plane of this angle to that of xz, then

$$\frac{x'-x}{R} = \cos\theta \quad \frac{d^2S}{du\,dv} = R^2\sin\theta;$$

$$\text{also } \cos\theta' = \sin\theta \sin v, \ \cos\theta'' = \sin\theta \cos v;$$

substitute and perform the integrations, and the expression is reduced to

$$\frac{4\pi k}{3} . a, \text{ thus } \frac{dQ_1}{dx} = \frac{dQ}{dx} - \frac{4\pi k}{3} . a;$$

$$\text{hence, } f = \frac{4\pi k}{3} . a - \frac{d}{dx}(V + Q);$$

$$\text{similarly, } g = \frac{4\pi k}{3} . b - \frac{d}{dy}(V + Q),$$

$$h = \frac{4\pi k}{3} . c - \frac{d}{dz}(V + Q).$$

Put $V + Q = U$ and observing that f, g, h are respectively proportional to a, b, c, it is easily seen that these equations may be put under the form

$$(1) \ldots\ldots\ldots \begin{cases} a + p\dfrac{dU}{dx} = 0 \\ b + p\dfrac{dU}{dy} = 0 \\ c + p\dfrac{dU}{dz} = 0; \end{cases}$$

and integrating the value of Q by parts and representing now by $\frac{d^2 S}{du\,dv} . \delta u\, \delta v$ the element of the surface of the whole body, as before of the sphere, so that θ, θ', θ'' are now the inclinations of the normal of that surface to the axes, we have

$$(2) \ldots\ldots Q = \int_u \int_v (a' \cos\theta + b' \cos\theta' + c' \cos\theta'') . \frac{d^2 S}{R\, du\, dv} - \phi,$$

putting for abridgment

$$\phi = \int_{x'} \int_{y'} \int_{z'} \frac{1}{R} \left\{ \frac{da'}{dx'} + \frac{db'}{dy'} + \frac{dc'}{dz'} \right\},$$

and a', b', c', for $k'a'$, $k'b'$ and $k'c'$.

Let U', p', denote the values of U, p, when x', y', z' are respectively substituted for x, y, z, then by the equation (1) we have

$$a' = -p'\frac{dU'}{dx'}, \quad b' = -p'\frac{dU'}{dy'}, \quad c' = -p'\frac{dU'}{dz'},$$

from whence equation (2) becomes.

$$(3) \ldots\ldots Q + \int_u \int_v \left(\cos\theta \,.\, \frac{dU'}{dx'} + \cos\theta' \frac{dU'}{dy'} + \cos\theta'' \frac{dU'}{dz'}\right) . \frac{p' d^2 S}{R\, du dv} + \phi = 0,$$

$$\text{where } \phi = -\int_{x'}\int_{y'}\int_{z'} \frac{1}{R}\left\{\frac{d\left(p'\frac{dU'}{dx'}\right)}{dx'} + \frac{d\left(p'\frac{dU'}{dy'}\right)}{dy'} + \frac{d\left(p'\frac{dU'}{dz'}\right)}{dz'}\right\},$$

$$\text{and } U' = Q' + V',$$

Q' and V' denoting the values of Q and V when x', y', z' replace x, y, z in those functions.

The equation (3) serves to determine Q, and then the required forces are known by the preceding article.

Cor. 1. The value of $Q = U - V$ is immediately deducible from that of U, which may be expressed by an equation to partial differences, thus;

By the known property of the function V, (Arts. 14. and 25), we have

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0,$$

or, denoting by Δ the operation here performed on V,

$$\Delta \,.\, V = 0,$$

$$\text{hence } \Delta \,.\, Q = \Delta \,.\, U.$$

Also $\Delta \dfrac{1}{R} = 0$; R being supposed never to vanish as in the double integral of equation (3), where R is the distance from any point in the surface of the body to one within it; that equation thus becomes

$$\Delta . Q + \Delta . \phi = 0.$$

But when R vanishes it is easily seen as in Art. 25. that if ρ' be any function of x', y', z', which becomes ρ when they are changed into x, y, z;

$$\text{then, } \Delta \int_{x'} \int_{y'} \int_{z'} \left(\frac{\rho'}{R}\right) = 4\pi\rho,$$

now this case happens in the function ϕ, whence we have

$$\Delta\phi = -4\pi \left\{\frac{d\left(p\dfrac{dU}{dx}\right)}{dx} + \frac{d\left(p\dfrac{dU}{dy}\right)}{dy} + \frac{d\left(p\dfrac{dU}{dz}\right)}{dz}\right\},$$

putting for $\Delta . Q$, $\Delta . \phi$ in the preceding equation their respective values, we get

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 4\pi \left\{\frac{d.p\dfrac{dU}{dx}}{dx} + \frac{d.p\dfrac{dU}{dy}}{dy} + \frac{d.p\dfrac{dU}{dz}}{dz}\right\}.$$

COR. 2. If the body is homogeneous and of uniform temperature, p will be constant and $\phi = 0$, since the last equation becomes in this case

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0,$$

hence by equation (3),

$$Q = -\int_u \int_v p \frac{d^2S}{R\,du\,dv} \left\{\cos\theta \frac{dU'}{dx'} + \cos\theta' \frac{dU'}{dy'} + \cos\theta'' \frac{dU'}{dz'}\right\};$$

hence, the magnetic actions of the system are exactly proportional to those that would be produced if the surface of the body were covered with a thin electrical stratum, the law of the accumulation of which at each point is expressed by the quantity between brackets in the value of Q.

(60). *To find the action of the same magnetic system in motion and under the influence of any forces; the body being supposed homogeneous and of uniform temperature.*

Retaining the notation and following the steps of the preceding article, if we suppose the force f to retain during the time t its initial value F, the equation $a' = Cf'$, or $a = Cf$ becomes in this case $a = C \cdot F$ at any given time, C being a function of the time which vanishes when $t = 0$, and representing it by $\psi(t)$, we have $a = \psi(t) \cdot F$ as the value which a would acquire on this supposition after the time t.

But since f is here variable, we must add to this primitive value the quantities similar to a which are generated from moment to moment by the increments of f, that is, if τ be the time corresponding to the variable force f_1;

$$\text{then, } a = F \cdot \psi(t) + \textstyle\int_\tau \psi(t-\tau) \cdot \dfrac{df_1}{d\tau}, \quad \text{from } \tau = 0 \text{ to } \tau = t$$

$$= -\textstyle\int_\tau f_1 \dfrac{d \cdot \psi(t-\tau)}{d\tau}$$

$$= \textstyle\int_\tau f_1 \cdot \psi'(t-\tau),$$

ψ' denoting the derived function from ψ.

$$\text{Similarly, } b = \textstyle\int_\tau f_1 \cdot \psi'(t-\tau),$$

$$c = \textstyle\int_\tau h_1 \cdot \psi'(t-\tau),$$

and putting

$$\frac{da'}{dx'} + \frac{db'}{dy'} + \frac{dc'}{dz'} = F'(t);$$

we get as before (since k is here constant),

$$Q = k \int_u \int_v (a' \cos\theta + b' \cos\theta' + c' \cos\theta'') \frac{d^2 S}{R\, du\, dv} - k \int_{x'} \int_{y'} \int_{z'} \frac{F'(t)}{R};$$

$$\text{hence } \Delta . Q = 4\pi k\, F(t),$$

$F(t)$ being the value of $F'(t)$ when x', y', z', are changed into x, y, z.

Also from the equations $f = \dfrac{4\pi k}{3} a - \dfrac{d}{dx}(V + Q)$,

&c.

$$\text{we get } \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = \frac{4\pi k}{3} . F(t) - \Delta . Q$$

$$= -\frac{8\pi k}{3} . F(t).$$

But from the values of a b and c, we have

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = \int_\tau \psi'(t-\tau) \left\{\frac{df_1}{dx} + \frac{dg_1}{dy} + \frac{dh_1}{dz}\right\};$$

hence, $F(t) = \dfrac{8\pi k}{3} \int_\tau F(\tau) . \psi'(t-\tau)$, from $\tau = 0$ to $\tau = t$,

which it is easily seen can only be satisfied by $F(t) = 0$;

$$\text{we have thus, } \frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0.$$

Again, if we put $\dfrac{da}{dy} - \dfrac{db}{dx} = F_1(t)$, we get in like manner

$$\frac{df}{dy} - \frac{dg}{dx} = \frac{4\pi k}{3} . F_1 t;$$

and therefore, $\psi(t) = \dfrac{4\pi k}{3} \int_\tau F_1(\tau) . \psi'(t-\tau)$,

$$\text{whence, } F_1(t) = 0, \text{ or } \frac{da}{dy} = \frac{db}{dx};$$

$$\text{similarly, } \frac{da}{dz} = \frac{dc}{dx}$$

$$\frac{db}{dz} = \frac{dc}{dy};$$

therefore, a, b, c must be the partial differential coefficients of a certain function U, the equation

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0$$

$$\text{becomes } \frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0;$$

and we get

$$Q = k\int_u\int_v \frac{d^2S}{R\,du\,dv}\left\{\frac{dU'}{dx'}\cos\theta + \frac{dU'}{dy'}\cos\theta' + \frac{dU'}{dz'}\cos\theta''\right\};$$

also, if a_1, V_1, Q_1 be the values of a, V, Q when $t = \tau$, we have

$$a = \int_\tau \left\{\frac{4\pi k}{3} a_1 - \frac{d}{dx}(V_1 + Q_1)\right\}.\psi'(t - \tau),$$

and corresponding equations for b and c; these equations are evidently the same as those derived by differentiating with respect to x, y and z the equation

$$U + \int_\tau \{V_1 + Q_1 - \frac{4\pi k}{3}.U_1\}.\psi'(t - \tau) = 0.$$

Remark. The evaluation of the magnetic actions on external points, whether the system be at rest or in motion, depends on the solution of the equations of this and the preceding article; M. Poisson to whom this theory is due, has made applications in his third memoir on Magnetism, to the case of a homogeneous sphere, hollow or solid, turning round

its axis, and influenced by terrestrial magnetism, and has shewn that the effect of the rotation is very nearly equivalent to that produced by a force normal to the plane passing through the axis in the direction of the influencing action of the earth, as observed previously by Mr Barlow; another application has been made to revolving plates, which illustrates the great difference between the actions of magnetic systems when in rest and motion, for M. Arago had previously discovered that bodies which have no magnetic power when at rest, yet when made rapidly to rotate round an axis, strongly influenced the magnetic needle in the direction of their motion.

It is also worthy of observation that the magnetic actions of bodies, whether at rest or in motion, may be assimilated to that of an electrical stratum, distributed solely on the surface, with a particular law of thickness, as appears from the value of Q in Arts. 58 and 59; Mr Barlow has made several experiments on the action of iron globes influenced by terrestrial magnetism, and has observed that the tangent of the dip of a small needle submitted to that action, is twice the tangent of the magnetic latitude of that needle with respect to an equatoreal circle drawn on the globe at right angles to the direction of terrestrial action, and that the law of force was the inverse cube of the distance. Now it is extremely easy to shew that a sphere under the influence of a remote body, and having originally equal quantities of positive and negative electricity, would produce exactly the same actions. (Vid. Arts. 19 and 31.)

END OF PART I.